

# Six-dimensional Supergravity and Projective Superfields

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## Abstract

We propose a superspace formulation of  $\mathcal{N} = (1, 0)$  conformal supergravity in six dimensions. The corresponding superspace constraints are invariant under super-Weyl transformations generated by a real scalar parameter. The known variant Weyl super-multiplet is recovered by coupling the geometry to a super-3-form tensor multiplet. Isotwistor variables are introduced and used to define projective superfields. We formulate a locally supersymmetric and super-Weyl invariant action principle in projective superspace. Some families of dynamical supergravity-matter systems are presented.

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## 1 Introduction

Recently, new off-shell superspace techniques have been developed to study general supergravity-matter systems with eight real supercharges in various space-time dimensions. These are based on the use of projective superspace, introduced in the 1980s by Karlhede, Lindström, and Roček to study rigid 4D,  $\mathcal{N} = 2$  supersymmetry [1, 2].<sup>1</sup> Analogously to harmonic superspace [6, 7], projective superspace is based on the extended superspace  $\mathbb{R}^{4|8} \times \mathbb{CP}^1$  where the projective coordinates are related to the  $SU(2)$  R-symmetry group of the extended supersymmetry algebra, an idea first introduced in the seminal paper [8].

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<sup>1</sup>See also [3] and references [4] and [5] for reviews on flat 4D,  $\mathcal{N} = 2$  projective superspace.

The first attempt to extend projective supermultiplets to curved supersymmetry was undertaken in 2007 in a study of matter couplings in 5D,  $\mathcal{N} = 1$  anti de-Sitter superspace [9]. This was subsequently adapted to supergravity in various dimensions in a series of papers, chronologically: 5D in [10, 11]; 4D in [12, 13]; 2D in [14]; and 3D in [15]. The formalism is based on two central ingredients: (i) a covariant geometric description in superspace of the supergravity multiplet; (ii) the existence of covariant projective multiplets, which are generalizations of the superconformal projective multiplets introduced by Kuzenko in [16, 17], and a locally supersymmetric and super-Weyl invariant action principle that is consistently defined on the curved geometry of ingredient (i).

In many respects, the curved projective superspace formalism has proven to resemble the covariant Wess-Zumino superspace approach to 4D,  $\mathcal{N} = 1$  supergravity [18]. However, while a prepotential description of 4D,  $\mathcal{N} = 2$  conformal supergravity was given in harmonic superspace in [19], its relationship to standard geometrical methods of curved superspace has not yet been elaborated in detail. A synthesis of curved harmonic and projective superspace could provide a coherent superspace description of 4D,  $\mathcal{N} = 2$  supergravity, along the lines of the Gates-Siegel approach to the 4D,  $\mathcal{N} = 1$  case [20]. Besides the calculational advantages this affords (*e.g.* background field quantization), such an understanding has applications in covariant string theory. These descriptions are (necessarily) closely related to the projective [21, 22] and harmonic superspace formalisms [23, 24]. A particularly relevant example is that of the pure spinor formalism [25] compactified on a K3 surface, where the physical state conditions on the unintegrated vertex operators are automatically formulated in terms of analyticity conditions in 6D,  $\mathcal{N} = (1, 0)$  projective superspace [26, 27]. Addition of the “missing” harmonics as non-minimal variables allows for a (simpler) description of the physical state conditions and the integrated vertex operators in harmonic superspace [27].

This paper is devoted to the continuation of the aforementioned program and to the demonstration that a projective superspace formalism can be efficiently implemented also in the case of six space-time dimensions. As a step toward the definition of six-dimensional curved projective multiplets, one first needs to identify a proper geometric description in superspace of off-shell,  $\mathcal{N} = (1, 0)$  supergravity. In a standard fashion, a starting point to describe off-shell supergravity systems is the coupling of the Weyl multiplet of conformal supergravity to matter compensators. This is possible both in components, through the superconformal tensor calculus techniques (see [28] for standard references), and in superspace. In components, the Weyl multiplet of 6D,  $\mathcal{N} = (1, 0)$  conformal

supergravity was constructed in reference [29]. To our knowledge, however, a geometric description of the Weyl multiplet in six dimensions, analogous in spirit to the 4D,  $\mathcal{N} = 2$  geometry of Howe [30], has hitherto not been fully developed.<sup>2</sup>

In this paper, we begin to fill this gap by presenting a superspace geometry suitable to the description of  $\mathcal{N} = (1, 0)$  conformal supergravity in six dimensions. Specifically, our geometry naturally describes the 40+40 components of the Weyl supermultiplet of [29] in superspace, in the form having the “matter” components of the multiplet described by an anti-self-dual 3-form  $W_{abc}^-$ , a positive-chirality spinor  $\chi^{\alpha i}$ , and a real scalar  $D$ . We will refer to this Weyl multiplet as the type-I multiplet. In reference [29], it was shown that there is a second 40+40 Weyl multiplet possessing as matter fields a scalar  $\sigma$ , a 2-form tensor  $B_{ab}$ , and a negative chirality spinor  $\psi_{\alpha i}$ ; we will refer to this as the type-II Weyl multiplet. Such a formulation is engineered by coupling the type-I multiplet to an on-shell tensor multiplet [32, 33] and solving for the type-I matter fields in terms of the fields of the tensor by using the equations of motion of the latter. The same mechanism can be used to describe the type-II Weyl multiplet in superspace as we will show by coupling the type-I superspace geometry to a tensor multiplet described in terms of a closed super 3-form (first introduced in the flat case in [34]).

Having constructed a superspace geometry suitable to the description of six-dimensional Weyl multiplets, the consistent definition of six-dimensional covariant projective superfields in this supergravity background proceeds exactly as in the lower-dimensional cases. In this paper, we will focus on such technical problems as the construction of the 6D,  $\mathcal{N} = (1, 0)$  multiplets, the projective action principle, and the analytic projection operator. We leave the applications of our results, some of which we set out in the Conclusion (section 4), for future investigation. Our hope is that the techniques we are starting to develop here will be of use not only in extending classic results (*e. g.* [35, 36]) but also newer ones which have arisen in the resurgence of interest in 6D,  $\mathcal{N} = (1, 0)$  supersymmetry and supergravity; see, for example, [37, 38, 39, 40, 41].

This work is organized into two parts with the supergeometrical part concentrated in section 2 and the projective superspace part presented in section 3. We begin the first part with the construction of the curved superspace geometry and give the dimension  $\leq \frac{3}{2}$  commutator algebra and torsion constraints in section 2.1. In section 2.2, we give

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<sup>2</sup> It is worth pointing out that a prepotential formulation of the Weyl multiplet was given by Sokatchev in 6D harmonic superspace [31].

the super-Weyl transformations compatible with this geometry and use them to elucidate the relation to the type-I multiplet of the superconformal tensor calculus. In section 2.3, we solve the Bianchi identities of a closed super-3-form in the type-I background and re-interpret the resulting supergravity-matter system in terms of the type-II Weyl multiplet. The second part begins with the construction of covariant projective superfields in six dimensions and the analytic projection operator in section 3.1. In section 3.2, we define the projective action principle, prove its consistency, and give families of examples of dynamical projective supergravity-matter systems. We conclude in section 4 with some reflection on our results and a description of future work and open problems. Our conventions are defined in appendix A and the requisite properties of the analytic projection operator are demonstrated in appendix B.

## 2 6D, $\mathcal{N} = (1, 0)$ Supergravity in Superspace

In this section, we present a new curved superspace geometry suitable to the description of  $\mathcal{N} = (1, 0)$  conformal supergravity in six dimensions. In the spirit of Howe and Tucker [42], we will see that the geometry is invariant under super-Weyl transformations generated by an unconstrained real scalar superfield. For this reason, the geometry will describe the 40+40 components of the type-I Weyl multiplet and, once coupled to a tensor multiplet super 3-form, the type-II Weyl multiplet. We refer the reader to the following list of references for previous work on flat superspace and multiplets in six dimensions: [33, 32, 43, 44, 45, 46, 47]. For the use of curved superspace to describe supergravity multiplets in six dimensions, see [48, 49, 50, 51, 52, 53, 54].

Our goal is to develop a formalism of differential geometry in a curved six-dimensional,  $\mathcal{N} = (1, 0)$  superspace  $\mathcal{M}^{6|8}$  that is locally parametrized by real bosonic ( $x^m$ ) and real fermionic ( $\theta_i^\mu$ ) coordinates

$$z^M = (x^m, \theta_i^\mu) , \quad m = 0, \dots, 3; 5, 6 , \quad \mu = 1, 2 , \quad i = \underline{1}, \underline{2} . \quad (2.1)$$

A natural condition on such a geometry is that it reduce to six-dimensional,  $\mathcal{N} = (1, 0)$  Minkowski superspace in the flat limit . Let us, to this end, recall that the spinor covariant derivatives  $D_\alpha^i$  associated with 6D,  $\mathcal{N} = (1, 0)$  Minkowski superspace satisfy the anti-commutation relations

$$\{D_\alpha^i, D_\beta^j\} = -2i \varepsilon^{ij} (\gamma^c)_{\alpha\beta} \partial_c . \quad (2.2)$$

An explicit realization of  $D_\alpha^i$  is given by the expression

$$D_\alpha^i = \frac{\partial}{\partial \theta_i^\alpha} + i(\gamma^b)_{\alpha\beta} \theta^{\beta i} \partial_b . \quad (2.3)$$

Given a superfield  $F$  of Grassmann parity  $\epsilon(F)$ , the conjugation rule of its covariant derivative is

$$\overline{(D_\alpha^i F)} = -(-)^{\epsilon(F)} D_{\alpha i} \bar{F} , \quad (2.4)$$

with  $\bar{F} := (F)^*$  the complex conjugate of  $F$ . Details of our notation and conventions are given in appendix A.

## 2.1 The Algebra of Covariant Derivatives

For our curved geometry, we choose the structure group to be  $\text{SO}(5, 1) \times \text{SU}(2)$ . The covariant derivative  $(\mathcal{D}_A) = (\mathcal{D}_a, \mathcal{D}_\alpha)$  expands as

$$\mathcal{D}_A = E_A + \Omega_A + \Phi_A , \quad (2.5)$$

where

$$E_A = E_A^M(z) \partial_M , \quad \Omega_A = \frac{1}{2} \Omega_A^{bc}(z) M_{bc} , \quad \Phi_A = \Phi_A^{ij}(z) J_{ij} , \quad (2.6)$$

denote the frame form, the spin connection, and the  $\text{SU}(2)$  connection, respectively. Here,  $\partial_M = \partial/\partial z^M$ ,  $M_{ab} = -M_{ba}$  is the Lorentz generator and  $J^{ij} = J^{ji}$  is the  $\text{SU}(2)$  R-symmetry generator. These are defined by their action on the spinor covariant derivatives as

$$[M_{ab}, \mathcal{D}_\gamma^k] = -\frac{1}{2} (\gamma_{ab})_\gamma{}^\delta \mathcal{D}_\delta^k , \quad [J^{ij}, \mathcal{D}_\gamma^k] = \varepsilon^{k(i} \mathcal{D}_\gamma^{j)} . \quad (2.7)$$

It follows that

$$[M_{ab}, \mathcal{D}_c] = 2\eta_{c[a} \mathcal{D}_{b]} . \quad (2.8)$$

The supergravity gauge group is generated by local transformations of the form

$$\delta_K \mathcal{D}_A = [K, \mathcal{D}_A] \quad \text{where} \quad K = K^C(z) \mathcal{D}_C + \frac{1}{2} K^{cd}(z) M_{cd} + \frac{1}{2} K^{kl}(z) J_{kl} , \quad (2.9)$$

with all the gauge parameters obeying natural reality conditions but otherwise arbitrary. Given a tensor superfield  $T(z)$ , its transformation rule is

$$\delta_K T = K T . \quad (2.10)$$

The covariant derivatives satisfy the (anti)commutation relations

$$[\mathcal{D}_A, \mathcal{D}_B] = T_{AB}{}^C \mathcal{D}_C + \frac{1}{2} R_{AB}{}^{cd} M_{cd} + F_{AB}{}^{kl} J_{kl} , \quad (2.11)$$

with  $T_{AB}{}^C$  the torsion,  $R_{AB}{}^{cd}$  the Lorentz curvature, and  $F_{AB}{}^{kl}$  the SU(2) R-symmetry field-strength. These tensor fields are related to each other by the Bianchi identities:

$$\sum_{[ABC]} [\mathcal{D}_A, [\mathcal{D}_B, \mathcal{D}_C]] = 0 . \quad (2.12)$$

To describe conformal supergravity, we impose conventional constraints on the torsion. In the six-dimensional case we are considering, they can be chosen to be

$$T_{\alpha\beta}{}^{ijc} = -2i\varepsilon^{ij}(\gamma^c)_{\alpha\beta} , \quad (\text{dimension-0}) \quad (2.13a)$$

$$T_{\alpha\beta k}{}^{ij\gamma} = 0 , \quad T_{\alpha b}{}^c = 0 , \quad (\text{dimension-}\frac{1}{2}) \quad (2.13b)$$

$$T_{ab}{}^c = 0 , \quad T_{a\beta(j}{}^{\beta}{}_{k)} = 0 . \quad (\text{dimension-1}) \quad (2.13c)$$

These constraints are similar to the four-dimensional,  $\mathcal{N} = 2$  superspace geometry of [30], formally identical to the five-dimensional conformal supergravity constraints of [11], and closely related to the six dimensional off-shell geometry of reference [48].

With the constraints so introduced, the solution of the Bianchi identities can be shown to imply that the torsion and curvature tensors in (2.11) are expressed in terms of a small number of mass-dimension-1 real tensor superfields  $C_{abc}$  and  $C_c{}^{ij}$ , and their covariant derivatives. The torsion component  $C_c{}^{ij} = C_c{}^{ji}$  is an iso-triplet and  $C_{abc} = W_{abc} + N_{abc}$  is a 3-form, which we split into anti-self-dual ( $W$ ) and self-dual ( $N$ ) parts.

In terms of these basic torsions, the graded commutation relations of the covariant derivatives are given by

$$\begin{aligned} \{\mathcal{D}_{\alpha i}, \mathcal{D}_{\beta j}\} &= 2i\varepsilon_{ij}(\gamma^a)_{\alpha\beta} \mathcal{D}_a + 2iC_{a ij}(\gamma^{abc})_{\alpha\beta} M_{bc} + 4i\varepsilon_{ij}W^{abc}(\gamma_a)_{\alpha\beta} M_{bc} \\ &\quad + 4i\varepsilon_{ij}N^{abc}(\gamma_a)_{\alpha\beta} M_{bc} - 6i\varepsilon_{ij}C_a{}^{kl}(\gamma^a)_{\alpha\beta} J_{kl} - \frac{8i}{3}N^{abc}(\gamma_{abc})_{\alpha\beta} J_{ij} , \end{aligned} \quad (2.14a)$$

$$[\mathcal{D}_{\gamma k}, \mathcal{D}_a] = -C_{kl}^b(\gamma_{ab})_{\gamma}{}^{\delta} \mathcal{D}_{\delta}^l + W_{abc}(\gamma^{bc})_{\gamma}{}^{\delta} \mathcal{D}_{\delta k} + N_{abc}(\gamma^{bc})_{\gamma}{}^{\delta} \mathcal{D}_{\delta k}$$

$$\begin{aligned}
& + \left[ \frac{i}{2}(\gamma_a)_{\gamma\delta} T_{bc}^{\delta} - i(\gamma_b)_{\gamma\delta} T_{ca}^{\delta} \right] M^{bc} \\
& + \left[ (\gamma_a)_{\gamma\delta} \mathcal{C}_k^{\delta ij} - 6\delta_k^i \mathcal{C}_a \gamma^j + 5(\gamma_a)_{\gamma\delta} \delta_k^i (\mathcal{C}^{\delta j} - \frac{1}{3}\mathcal{W}^{\delta j}) \right] J_{ij} ,
\end{aligned} \tag{2.14b}$$

where  $T_{ab}^{\gamma}$ ,  $\mathcal{C}_i^{\alpha}$ , and  $\mathcal{W}_i^{\alpha}$  are defined below (c.f. eq. 2.17, 2.15a, and 2.15c, resp.). The dimension-1 superfields  $C_{a ij}$ ,  $W^{\alpha\beta} := \frac{1}{6}W_{abc}(\tilde{\gamma}^{abc})^{\alpha\beta}$ , and  $N_{\alpha\beta} := \frac{1}{6}N_{abc}(\gamma^{abc})_{\alpha\beta}$  satisfy additional constraints which follow from the Bianchi identities. To display the content of these constraints more clearly, we first define their Lorentz- and isospin-irreducible components

$$\mathcal{D}_{\gamma k} C_{a ij} =: \mathcal{C}_{a \gamma k ij} + (\gamma_a)_{\gamma\delta} \mathcal{C}_{ijk}^{\delta} + \varepsilon_{k(i} \mathcal{C}_{a \gamma j)} + \varepsilon_{k(i} (\gamma_a)_{\gamma\delta} \mathcal{C}_{j)}^{\delta} , \tag{2.15a}$$

$$\mathcal{D}_{\gamma k} N_{\alpha\beta} =: \mathcal{N}_{\gamma k \alpha\beta} + \tilde{\mathcal{N}}_{\gamma k \alpha\beta} , \tag{2.15b}$$

$$\mathcal{D}_{\gamma k} W^{\alpha\beta} =: \mathcal{W}_{\gamma k}^{\alpha\beta} + \delta_{\gamma}^{(\alpha} \mathcal{W}_k^{\beta)} . \tag{2.15c}$$

Multiple isospin indices are fully symmetrized as are multiple Lorentz indices of the same height (except for the case  $\tilde{\mathcal{N}}$ , which has a part proportional to a  $\gamma$ -matrix; c.f. eq. 2.16), Lorentz indices at different heights have had their traces removed, and fields with both vector and spinor indices are  $\gamma$ -traceless. These properties are reflected in their explicit forms as solutions to the Bianchi identities:

$$\begin{aligned}
\mathcal{C}_{a \gamma k ij} &= 0 , & \mathcal{N}_{\gamma k \alpha\beta} &= 0 , \\
\mathcal{C}_{ijk}^{\delta} &= -\frac{1}{6}(\tilde{\gamma}^b)^{\delta\beta} \mathcal{D}_{\beta(k} C_{bij)} , & \tilde{\mathcal{N}}_{\gamma k \alpha\beta} &= -\frac{3}{4}(\gamma^a)_{\gamma(\alpha} \mathcal{C}_{a \beta)k} , \\
\mathcal{C}_{a \beta j} &= \frac{2}{3}\Pi_{a\beta}^{\gamma} \mathcal{D}_{\gamma} C_{a ij} , & \mathcal{W}_{\gamma k}^{\alpha\beta} &= \mathcal{D}_{\gamma k} W^{\alpha\beta} - \frac{2}{5}\delta_{\gamma}^{(\alpha} \mathcal{D}_{\delta k} W^{\beta)\delta} , \\
\mathcal{C}^{\gamma k} &= -\frac{1}{9}\mathcal{D}_{\delta l} C^{\delta \gamma lk} , & \mathcal{W}^{\alpha i} &= \frac{2}{5}\mathcal{D}_{\beta}^i W^{\beta\alpha} .
\end{aligned} \tag{2.16}$$

Here,  $\Pi_{a\alpha}^{b\beta} = \delta_a^b \delta_{\alpha}^{\beta} + \frac{1}{6}(\gamma_a \tilde{\gamma}^b)_{\alpha}^{\beta}$  is the projector onto the  $\gamma$ -traceless subspace (*i.e.*  $(\tilde{\gamma}^a)^{\gamma\alpha} \Pi_{a\alpha}^{b\beta} = 0 = \Pi_{a\alpha}^{b\beta} (\gamma_b)_{\beta\gamma}$ ). We note that, of the two components of  $\mathcal{D}N$ , one vanishes and the other is related to an irreducible part of  $\mathcal{D}C$ .

Finally, the dimension- $\frac{3}{2}$  torsion is given in terms of the remaining fields as

$$T_{ab}^{\gamma k} = \frac{i}{2}(\gamma_{ab})_{\beta}^{\delta} \mathcal{W}_{\delta}^k{}^{\beta\gamma} + \frac{7i}{4}(\tilde{\gamma}_{[a})^{\gamma\delta} \mathcal{C}_{b]}{}_{\delta}^k + i(\gamma_{ab})_{\delta}^{\gamma} [\mathcal{C}^{\delta k} - \frac{1}{6}\mathcal{W}^{\delta k}] . \tag{2.17}$$

With this, the dimension-1 and dimension- $\frac{3}{2}$  commutators are completely specified. It has been verified that the Bianchi identities are satisfied up to and including dimension  $\frac{3}{2}$ . Further details of the geometry are not required for the purposes of this paper and will be expounded upon elsewhere.



## 2.2 Super-Weyl Transformations and the Type-I Weyl Multiplet

A short calculation shows that the constraints (2.13a)—(2.13c) are invariant under arbitrary super-Weyl transformations defined by

$$\delta \mathcal{D}_{\alpha i} = \frac{1}{2} \sigma \mathcal{D}_{\alpha i} - 2(\mathcal{D}_{\beta i} \sigma) M_{\alpha}^{\beta} + 4(\mathcal{D}_{\alpha}^j \sigma) J_{ij} , \quad (2.18a)$$

$$\delta \mathcal{D}_a = \sigma \mathcal{D}_a - \frac{i}{2} (\mathcal{D}^k \sigma) \tilde{\gamma}_a \mathcal{D}_k - (\mathcal{D}^b \sigma) M_{ab} - \frac{i}{8} (\mathcal{D}^i \tilde{\gamma}_a \mathcal{D}^j \sigma) J_{ij} , \quad (2.18b)$$

where the parameter  $\sigma(z)$  is a real, unconstrained superfield. The components of the dimension-1 torsion are required to transform as

$$\delta C_{a ij} = \sigma C_{a ij} + \frac{i}{8} (\mathcal{D}_i \tilde{\gamma}_a \mathcal{D}_j \sigma) , \quad (2.19a)$$

$$\delta W_{abc} = \sigma W_{abc} , \quad (2.19b)$$

$$\delta N_{abc} = \sigma N_{abc} - \frac{i}{32} (\mathcal{D}^k \tilde{\gamma}_{abc} \mathcal{D}_k \sigma) . \quad (2.19c)$$

The transformations of  $C_{a ij}$  and  $N_{abc}$  contain in-homogeneous terms which can be used to gauge away many of their components. The anti-self-dual 3-form  $W_{abc}$  transforms homogeneously and represents a superspace generalization of the Weyl tensor. It can be shown that, by properly choosing a Wess-Zumino gauge for our superspace geometry, the surviving physical components embedded in the geometry contain the SU(2) field-strength, the gravitino curl, an anti-self-dual auxiliary 3-form of mass dimension-1, an auxiliary spinor of positive chirality of mass dimension- $\frac{3}{2}$ , the Weyl tensor, and a real auxiliary scalar field of mass-dimension-2. The resulting multiplet describes the (40+40)-component Weyl supermultiplet [29]

$$e_a^m, \psi_m^{\alpha i}, \Phi_a^{ij}, W_{abc}^-, \chi^{\alpha i}, D , \quad (2.20)$$

to which we will refer as the type-I multiplet. Here,  $e_a^m$  is the sechsbein,  $\psi_m^{\alpha i}$  the gravitino,  $\Phi_a^{ij}$  the SU(2) connection, and  $W_{abc}^-$ ,  $\chi^{\alpha i}$ ,  $D$  are the “matter” fields. Here, the component gauge fields and the gravitino are related to the  $\theta = 0$  components of the supersechsbein and superconnections whereas the matter fields of the Weyl multiplet arise in our geometry as components of the Weyl superfield:  $W_{abc}^- = W_{abc}|_{\theta=0}$ ,  $\chi^{\alpha i} = \mathcal{W}^{\alpha i}|_{\theta=0}$  and  $D = \mathcal{D}_{\alpha i} \mathcal{W}^{\alpha i}|_{\theta=0}$ . As originally defined [29], this Weyl multiplet contains an additional dilatation gauge field  $b_m(x)$  but this degree of freedom is pure gauge and one can choose to work in the gauge in which it vanishes. Such a gauge arises naturally

in the superspace geometry we have introduced here. This situation is similar to the 5D conformal supergravity in superspace described in [11] and to Grimm's formulation of 4D supergravity [55], as explained in detail in [13]. In these superspace treatments, as with our geometry, the  $b_m$  field does not arise.

### 2.3 The Tensor Multiplet and the Type-II Weyl Multiplet

There is a second formulation of the Weyl supermultiplet in which the anti-self-dual 3-form  $W$ , auxiliary positive chirality spinor  $\chi$ , and auxiliary scalar  $D$ , are replaced by a tensor multiplet consisting of a propagating scalar  $\sigma$ , a gauge 2-form tensor  $B$ , and a negative chirality tensorino  $\chi$  [29]. This alternate formulation, to which we will refer as the type-II Weyl multiplet, plays an important role in six-dimensional supergravity since it is the one that, within the superconformal tensor calculus approach, can be consistently used to construct actions for general matter-coupled supergravity systems. (See, for example, [38] for a recent discussion of six-dimensional Poincaré supergravity obtained by coupling the type-II Weyl multiplet to a linear multiplet.) In this subsection, following the same logic used in the component case, we work out the superspace version of the type-II formulation by coupling the type-I formulation to a tensor multiplet [32, 33]. In flat space, the tensor multiplet has been constructed as a closed 3-form in superspace in [34]. It is natural to formulate the consistent curved tensor multiplet constraints extending such a construction to our curved superspace geometry. To this end, we must work out the mass dimension  $\leq 3$  part of the 3-form Bianchi identities in the supergravity background.

The super-3-form  $H$  can be written in local coordinates as

$$H = \frac{1}{3!} dZ^{M_3} dZ^{M_2} dZ^{M_1} H_{M_1 M_2 M_3} = \frac{1}{3!} E^{A_3} E^{A_2} E^{A_1} H_{A_1 A_2 A_3} . \quad (2.21)$$

This form is closed,  $dH = 0$ , iff its components satisfy the Bianchi identities

$$\frac{1}{3!} \mathcal{D}_{[B} H_{A_1 A_2 A_3]} - \frac{1}{2! \cdot 2!} T_{[B A_1]}^C H_{C | A_2 A_3]} = 0 . \quad (2.22)$$

The dimension-2 condition is consistent with the constraint

$$H_{\alpha i \beta j \gamma k} = 0 , \quad (2.23)$$

provided that

$$H_{\alpha i \beta j c} = 2i \varepsilon_{ij} (\gamma_c)_{\alpha \beta} \Phi , \quad (2.24)$$

where  $\Phi$  is an arbitrary real scalar superfield. Next, the dimension- $\frac{5}{2}$  identity is solved by

$$H_{\alpha i b c} = -(\gamma_{bc})_{\alpha}{}^{\beta} \mathcal{D}_{\beta i} \Phi . \quad (2.25)$$

Finally, the dimension-3 identity gives the expression for the 3-form  $H_{abc} = H_{abc}^{+} + H_{abc}^{-}$

$$H_{abc}^{+} = \left( \frac{i}{8} \mathcal{D}^k \tilde{\gamma}_{abc} \mathcal{D}_k - 16 N_{abc} \right) \Phi , \quad (2.26a)$$

$$H_{abc}^{-} = -16 W_{abc} \Phi , \quad (2.26b)$$

divided here into its self-dual and anti-self-dual parts. Additionally, it implies that  $\Phi$  satisfies

$$\mathcal{D}_{(i} \tilde{\gamma}^a \mathcal{D}_{j)} \Phi + 16 i C_{ij}^a \Phi = 0 . \quad (2.27)$$

This constraint is super-Weyl invariant iff  $\Phi$  has scaling-dimension equal to 2, that is,  $\delta \Phi = 2\sigma \Phi$ . It is the curved-space analogue of the flat space constraint  $D_{\alpha}^{(i} D_{\beta}^{j)} \Phi = 0$  which describes the tensor multiplet consisting of a scalar<sup>3</sup>  $\sigma(x) \sim \Phi(z)|_{\theta=0}$ , a tensorino  $\psi_{\alpha i}(x) \sim D_{\alpha i} \Phi(z)|_{\theta=0}$ , and a self-dual 3-form field-strength  $h_{abc}^{+}(x) \sim D^k \tilde{\gamma}_{abc} D_k \Phi(z)|_{\theta=0}$  [34]. Indeed, using (2.25)–(2.26b), one derives the Bianchi identity

$$\frac{1}{3!} \mathcal{D}_{[a} H_{bcd]} - \frac{1}{2! \cdot 2!} T_{[ab}{}^{\gamma k} H_{cd] \gamma k} = 0 , \quad (2.28)$$

which implies that, up to spinorial torsion terms, the 3-form superfield  $H_{abc} \sim \mathcal{D}_{[a} B_{bc]}$  is locally exact. Finally, we note that the constraint (2.27) puts the tensor multiplet on-shell. This is most easily checked by taking the flat-space limit,  $D_{\alpha}^{(i} D_{\beta}^{j)} \Phi = 0$ , and showing that it implies, for example,  $\partial^a \partial_a \Phi = 0$ . In the supergravity case, the equations are covariantized by the supergravity fields which provide additional interaction terms.

In fact, it can be shown that the constraint (2.27) is equivalent to the condition

$$\mathcal{D}_{\alpha(i} V_{j)}^{\beta} - \frac{1}{4} \delta_{\alpha}^{\beta} \mathcal{D}_{\gamma(i} V_{j)}^{\gamma} = 0 , \quad (2.29)$$

on a spinor potential superfield  $V^{\alpha i}$ , provided we identify

$$\Phi = \mathcal{D}_{\alpha i} V^{\alpha i} . \quad (2.30)$$

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<sup>3</sup>To follow the nomenclature normally used in the literature for the component fields of the tensor multiplet, we call the lowest component field  $\sigma(x)$ . Context should serve to distinguish this component from the super-Weyl parameter superfield  $\sigma(z)$ .

In flat superspace, this multiplet was first introduced by Sokatchev in [31]. It is straightforward to verify that the new constraint is invariant under super-Weyl transformations iff  $V$  is a super-Weyl tensor of scaling dimension  $-\frac{3}{2}$ :  $\delta_\sigma V^{\alpha i} = \frac{3}{2}\sigma V^{\alpha i}$ . Furthermore, it is invariant under a gauge transformation so that, in the components of this new multiplet, the superfield 3-form field-strength  $H_{abc} = H_{abc}^+ + H_{abc}^-$  is replaced with the exterior derivative of its gauge 2-form potential  $B_{ab} \sim \mathcal{D}_i \tilde{\gamma}_{ab} V^i$  [31]. It is non-trivial that  $\Phi$ , as defined in (2.30), is a super-Weyl tensor of scaling dimension -2.

As first pointed out in reference [29], provided that the scalar component  $\sigma(x) \sim \Phi|_{\theta=0}$  is everywhere non-vanishing, the equations of motion can be solved for the components  $\{W^-, \chi, D\}$  in terms of the components  $\{\sigma, B, \psi\}$ . The result is a description in terms of the components of the type-II Weyl multiplet [29]

$$e_a^m, \psi_m^{\alpha i}, \Phi_a^{ij}, \sigma, \psi_{\alpha i}, B_{ab} . \quad (2.31)$$

This formulation can be interpreted as arising by taking the  $(40 + 40)$ -component type-I multiplet, coupling to the  $11 + 8$  fields  $\{\sigma, B, \psi\}$ , and then imposing the  $11 + 8$  degrees of freedom of the equations of motion as constraints. In this interpretation, the tensor supermultiplet does not add any degrees of freedom to the type-I multiplet overall. In our superspace language, assuming the superfield  $\Phi(z) \neq 0$  is everywhere non-vanishing, this is equivalent to solving for the dimension-1 torsion superfields of the type-I geometry in terms of the tensor multiplet 3-form superfields.

This suggests a second mechanism to remove the newly added tensor-multiplet degrees of freedom: Whenever the scalar field in the superfield  $\Phi$  is nowhere-vanishing on the body of the supermanifold, it is evidently possible to use the super-Weyl parameter to gauge  $\Phi \rightarrow 1$ . Equation (2.27) then reduces to  $C_a^{ij} = 0$  and equations (2.23)–(2.26a) become

$$H_{\alpha i \beta j \gamma k} = 0 , \quad H_{\alpha i \beta j c} = 2i\varepsilon_{ij}(\gamma_c)_{\alpha\beta} , \quad H_{\alpha i b c} = 0 , \quad H_{abc}^+ = -16N_{abc} , \quad H_{abc}^- = -16W_{abc} . \quad (2.32)$$

This super-Weyl gauge corresponds to further strengthening the second conventional constraint in equation (2.13c) by imposing  $T_{a\beta(jk)}^\gamma \rightarrow 0$ . (Equivalent observations were made already in reference [54].) The residual Weyl transformations are constrained by (2.19a) to satisfy  $\mathcal{D}_{\alpha(i}\mathcal{D}_{\beta j)}\sigma = 0$ . There is still enough of the Weyl parameter left to remove all of the remaining components of the self-dual field  $N_{\alpha\beta}$ . This leaves only the anti-self-dual part  $W^{\alpha\beta}$  which, in this formulation, carries all of the off-shell degrees of freedom of the type-I multiplet.

It is interesting to note that in five dimensions there is a mechanism similar to the one just described to formulate a variant Weyl multiplet. In fact, by coupling the five-dimensional Weyl multiplet to an abelian vector multiplet constrained to satisfy the curved Chern-Simons equation of motion, one can solve it for the auxiliary fields of the standard Weyl multiplet and end up with the so-called dilaton-Weyl multiplet [56, 57]. See reference [11] for a description of this mechanism in superspace.

We conclude this section by comparing the six-dimensional variant to the lower-dimensional cases. In  $D = 4$  and  $5$ , vector multiplets with eight supercharges are of primary importance for conformal supergravity since they possess a scalar field as their lowest component. For this reason, off-shell vector multiplets are the most natural conformal compensators in  $4D$ ,  $\mathcal{N} = 2$  and  $5D$ ,  $\mathcal{N} = 1$  supergravity and, once coupled to the Weyl multiplet, give rise to the so-called minimal multiplets [30, 58, 59]. In six dimensions, on the other hand, the lowest component of an off-shell vector multiplet is a positive-chirality Weyl spinor. In superspace, the  $6D$  off-shell vector multiplet is described by a dimension- $\frac{3}{2}$  superfield-strength  $F_i^\alpha$  constrained by [60]

$$\mathcal{D}_\alpha^{(i} F^{\beta j)} - \frac{1}{4} \delta_\alpha^\beta \mathcal{D}_\gamma^{(i} F^{\gamma j)} = 0 \quad \text{and} \quad \mathcal{D}_\alpha^i F_i^\alpha = 0 \quad (2.33)$$

which, compared with the tensor multiplet constraint (2.29), is missing the scaling-dimension-2 scalar superfield (2.30).<sup>4</sup> Due to the differences just mentioned, there is no direct analogue of the minimal multiplet in six dimensions. In some respects, the  $6D$  tensor multiplet closely mimics features of the lower-dimensional vector multiplets. It has a scalar that naturally plays the role of a dilaton but the crucial difference is that the  $6D$  tensor multiplet is on-shell.

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<sup>4</sup>Note that the tensor multiplet defined by the constraint (2.29), which is the first of the two vector multiplet constraints in (2.33), has on-shell physical fields while the vector multiplet is off-shell. The  $V^{\alpha i}$ -multiplet includes the following physical fields: A 2-form gauge potential  $B^{ab} \sim (\gamma^{ab})^\alpha{}_\beta D_{\alpha i} V^{\beta i}|_{\theta=0}$ ; a scalar dilaton  $\sigma \sim D_{\alpha i} V^{\alpha i}|_{\theta=0}$ ; and a fermion  $\chi_{\alpha i} \sim D_{\alpha i} D_\beta^j V_j^\beta|_{\theta=0}$ . The vector multiplet, on the other hand, consists of the following physical fields: A gaugino  $\lambda^{\alpha i} \sim F^{\alpha i}|_{\theta=0}$ ; a gauge field strength  $F^{ab} \sim (\gamma^{ab})^\alpha{}_\beta D_{\alpha i} F^{\beta i}|_{\theta=0}$ ; and an auxiliary iso-triplet  $Y^{ij} \sim D_\alpha^{(i} F^{\alpha j)}$ . One can check that the components of the vector multiplet that are also components of the  $V^{\alpha i}$  superfield are pure gauge in the latter case. On the other hand, the physical fields of  $V^{\alpha i}$  that are responsible for putting the multiplet on-shell are precisely those killed by the second constraint in (2.33).

### 3 Six-dimensional Curved Projective Superspace

Covariant projective supermultiplets have been used recently to efficiently describe matter couplings in extended supergravity. This was first done in five dimensions [10, 11], then applied to the four-dimensional case [12, 13], and recently extended to two [14] and then three [15] dimensions. In this section, we continue this program by showing that the existence of covariant projective supermultiplets is consistent with the geometry of section 2. (Projective superfields in flat 6D,  $\mathcal{N} = (1, 0)$  Minkowski superspace were first introduced in [46] and further studied in [47].) We then conclude with a presentation of a locally supersymmetric and super-Weyl invariant action principle.

#### 3.1 6D, $\mathcal{N} = (1, 0)$ Covariant Projective Superfields

In defining curved projective multiplets, we follow the same procedure that has been successfully developed in the  $2 \leq D \leq 5$  supergravity cases [10, 11, 12, 13, 14, 15]. We start by introducing an *isotwistor* variable  $v^i = (v^1, v^2) \in \mathbb{C}^2 \setminus \{0\}$ , defined to be inert under the action of the supergravity structure group:  $[M_{ab}, v^i] = [J_{kl}, v^i] = 0$ . Using this isotwistor, we define the covariant derivatives

$$\mathcal{D}_\alpha^{(1)} := v_i \mathcal{D}_\alpha^i . \quad (3.1)$$

Note that the  $\mathcal{D}_\alpha^{(1)}$  derivative is homogeneous of degree one in  $v^i$ . Our curved superspace is then extended to  $\mathcal{M}^{6|8} \times \mathbb{CP}^1$ , with the isotwistor variable interpreted as providing homogeneous coordinates  $[v^1 : v^2]$  of the complex projective line. Tensor superfields on this extension are called *isotwistor superfields*. A weight- $n$  isotwistor superfield  $U^{(n)}(z, v)$  is defined to be holomorphic on an open domain of  $\mathbb{C}^2 \setminus \{0\}$  with respect to the homogeneous coordinates  $v^i$  for  $\mathbb{CP}^1$  and is characterized by the conditions:

- (i) It is a homogeneous function of  $v$  of degree  $n$ , that is,

$$U^{(n)}(z, cv) = c^n U^{(n)}(z, v) , \quad c \in \mathbb{C}^* , \quad (3.2)$$

- (ii) The supergravity gauge transformations act on  $U^{(n)}$  as follows:

$$\delta_K U^{(n)} = \left( K^C \mathcal{D}_C + \frac{1}{2} K^{cd} M_{cd} + K^{kl} J_{kl} \right) U^{(n)} , \quad (3.3a)$$

$$J_{kl} U^{(n)}(v) = -\frac{1}{(v, u)} \left( v_{(k} v_{l)} u^i \frac{\partial}{\partial v^i} - n v_{(k} u_{l)} \right) U^{(n)}(v) , \quad (3.3b)$$

where

$$(v, u) := v^i u_i, \quad \delta_j^i = \frac{1}{(v, u)} (v^i u_j - v_j u^i) . \quad (3.4)$$

The auxiliary variable  $u_i$  is constrained by  $(v, u) \neq 0$  but is otherwise completely arbitrary. By definition,  $U^{(n)}$  is a function only of  $v$  and not  $u$ ; the same should be true for its variation. Indeed, due to (3.2), the superfield  $(J_{kl}U^{(n)})$  can be seen to be independent of  $u_i$  even though the transformations in (3.3b) explicitly depend on it.

With the definitions (i) and (ii) assumed, the set of isotwistor superfields is closed under products and the action of the  $\mathcal{D}_\alpha^{(1)}$  derivative. More precisely, given weight- $m$  and weight- $n$  isotwistor superfields  $U^{(m)}$  and  $U^{(n)}$ , the superfield  $(U^{(m)}U^{(n)})$  is a weight- $(m+n)$  isotwistor superfield and the superfield  $(\mathcal{D}_\alpha^{(1)}U^{(n)})$  is a weight- $(n+1)$  isotwistor superfield. Note that, as implicitly indicated in (3.3a), general isotwistor superfields are not restricted to be Lorentz scalar. Ultimately, the use of the extra isotwistor variable should be interpreted as an efficient way to deal with superfields that are (in general, infinite-dimensional) representations of the  $SU(2)$  group; see [12, 13] for more details.

The most important property of isotwistor superfields is that the anti-commutator of  $\mathcal{D}_\alpha^{(1)}$  covariant derivatives is zero when acting on a Lorentz scalar, isotwistor superfield  $U^{(n)}$ . In fact, from (2.14a), one obtains the anti-commutation relation

$$\{\mathcal{D}_\alpha^{(1)}, \mathcal{D}_\beta^{(1)}\} = -8iC_{\gamma(\alpha}^{(2)}M_{\beta)}^\gamma - 16iN_{\alpha\beta}J^{(2)}, \quad (3.5)$$

where we have defined

$$C_{\alpha\beta}^{(2)} := v_i v_j C_{\alpha\beta}^{ij} \quad \text{and} \quad J^{(2)} := v^i v^j J_{ij}. \quad (3.6)$$

The  $SU(2)$  generators appear in the previous algebra only in the combination defined by  $J^{(2)}$  which can easily be shown to vanish when acting on general isotwistor superfields  $J^{(2)}U^{(n)} = 0$ . If one imposes that  $U^{(n)}$  be a Lorentz scalar, then

$$\{\mathcal{D}_\alpha^{(1)}, \mathcal{D}_\beta^{(1)}\}U^{(n)} = 0. \quad (3.7)$$

We define a weight- $n$ , covariant projective superfield  $Q^{(n)}(z, v)$  to be an isotwistor superfield (*i.e.* satisfying (i) and (ii)) constrained by the analyticity condition

$$\mathcal{D}_\alpha^{(1)}Q^{(n)} = 0. \quad (3.8)$$

The consistency of the previous constraint is guaranteed by equation (3.7) which takes the form of an integrability condition for the analyticity constraint (3.8).

In conformal supergravity, the important issue is whether the projective multiplets can be made to vary consistently under the super-Weyl transformations. Suppose we are given a weight- $n$ , projective superfield  $Q^{(n)}$  that transforms homogeneously:  $\delta_\sigma Q^{(n)} \propto \sigma Q^{(n)}$ . Its transformation law is, then, uniquely fixed to be

$$\delta_\sigma Q^{(n)} = 2n\sigma Q^{(n)} , \quad (3.9)$$

by imposing super-Weyl invariance of the constraint (3.8).

Given a projective multiplet  $Q^{(n)}$ , its complex conjugate is not covariantly analytic. However, one can introduce a generalized analyticity-preserving conjugation  $Q^{(n)} \rightarrow \check{Q}^{(n)}$ , defined as

$$[Q^{(n)}(v)]^\vee \equiv \bar{Q}^{(n)}(\bar{v} \rightarrow \check{v}) , \quad \check{v} = i\sigma_2 v , \quad (3.10)$$

with  $\bar{Q}^{(n)}(\bar{v})$  the complex conjugate of  $Q^{(n)}(v)$ . It follows that  $\check{\check{Q}}^{(n)} = (-1)^n Q^{(n)}$  so that real supermultiplets can be consistently defined when  $n$  is even. The superfield  $\check{Q}^{(n)}$  is called the *smile-conjugate* of  $Q^{(n)}$ . Note that, geometrically, this conjugation is a composition of complex conjugation and the antipodal map on  $\mathbb{CP}^1$ . A fundamental property is that

$$[\mathcal{D}_\alpha^{(1)} Q^{(n)}]^\vee = (-1)^{\epsilon(Q^{(n)})} \mathcal{D}_\alpha^{(1)} \check{Q}^{(n)} , \quad (3.11)$$

implying that the analytic constraint (3.8) is invariant under smile conjugation.

A simple class of 6D projective superfields is defined as  $G^{(m)}(z, v) = v_{i_1} \cdots v_{i_m} G^{i_1 \cdots i_m}(z)$ . These are constructed in terms of the completely symmetric isotensor superfields  $G^{i_1 \cdots i_m}(z) = G^{(i_1 \cdots i_m)}(z)$  and describe regular holomorphic tensor fields on the complex projective space  $\mathbb{CP}^1$  parametrized by the homogeneous coordinates  $v^i$ . Provided that the  $SU(2)$  transformation rule of  $G^{i_1 \cdots i_m}$  is the standard one

$$J_{kl} G^{i_1 \cdots i_m} = \delta_{(k}^{(i_1} G_{l)}^{i_2 \cdots i_m)} , \quad (3.12)$$

the superfield  $G^{(m)}$  satisfies all the conditions to be isotwistor. Moreover, the analyticity condition  $\mathcal{D}_\alpha^{(1)} G^{(m)} = 0$  is equivalent to the following constraint on  $G^{i_1 \cdots i_m}$ :

$$\mathcal{D}_\alpha^{(j)} G^{i_1 \cdots i_m} = 0 . \quad (3.13)$$



This constraint is consistent with the super-Weyl transformation rule  $\delta_\sigma G^{i_1 \dots i_m} = 2m\sigma G^{i_1 \dots i_m}$ . When  $m = 2p$ , one can further constrain  $G^{(2p)}$  to be smile-real which is equivalent to the condition  $\overline{(G^{i_1 \dots i_{2p}})} = G_{i_1 \dots i_{2p}}$ . This kind of multiplet is known in  $4D, \mathcal{N} = 2$  supersymmetry literature as an  $\mathcal{O}(2p)$ -multiplet. It is a generalization of the well-known linear multiplet  $\overline{G^{i\bar{j}}} = G_{i\bar{j}}$  that has  $p = 1$ ; for an incomplete list of references see [61, 62, 63, 2, 64, 65]. Note that when  $m = 1$ ,  $G^{(1)} = v_i q^i$ , the (necessarily complex) superfield  $q^i$  satisfies  $\mathcal{D}_\alpha^{(i} q^{j)} = 0$  and describes a six-dimensional extension of the Fayet-Sohnius hypermultiplet [66]. It is necessarily on-shell as a consequence of the impossibility of adding a central charge to the  $6D, \mathcal{N} = (1, 0)$  algebra.

Instead of the homogeneous coordinates  $[v^\perp : v^2]$ , it is often useful to work with an inhomogeneous local complex variable  $\zeta$  that is invariant under arbitrary projective rescalings  $v^i \rightarrow c v^i$ , with  $c \in \mathbb{C}^*$ . In such an approach, one should replace  $Q^{(n)}(z, v)$  with a new superfield  $Q^{[n]}(z, \zeta) \propto Q^{(n)}(z, v)$ , where  $Q^{[n]}(z, \zeta)$  is holomorphic with respect to  $\zeta$ . Its explicit definition depends on the supermultiplet under consideration. The space  $\mathbb{CP}^1$  can naturally be covered by two open charts in which  $\zeta$  can be defined, and the simplest choice is: (i) the north chart characterized by  $v^\perp \neq 0$ ; (ii) the south chart with  $v^2 \neq 0$ . In the projective superspace literature, the classification of multiplets normally proceeds by restricting to the north chart and depends on the pole structure of Laurent expansion in  $\zeta$ . Analogously to the curved cases in dimensions  $2 \leq D \leq 5$  [10, 11, 12, 13, 14, 15], six-dimensional projective superfields generically possess an infinite number of standard superfields. As an example, consider off-shell charged hypermultiplets. In projective superspace these have a natural description in terms of the so-called arctic superfield: A *weight- $n$  polar multiplet* is described in terms of *arctic* superfields  $\Upsilon^{(n)}(z, v)$ , and their *antarctic* smile-conjugates  $\check{\Upsilon}^{(n)}(z, v)$ . By definition,  $\Upsilon^{(n)}$  is a projective superfield that is well-defined in the whole north chart of  $\mathbb{CP}^1$  (conversely  $\check{\Upsilon}^{(n)}(z, v)$  is well-defined in the whole south chart). In the north chart,  $\Upsilon^{(n)} = (v^\perp)^n \Upsilon^{[n]}$  and  $\check{\Upsilon}^{(n)} = (v^2)^n \check{\Upsilon}^{[n]} = (v^\perp)^n \zeta^n \check{\Upsilon}^{[n]}$  are represented as

$$\Upsilon^{[n]}(z, \zeta) = \sum_{k=0}^{\infty} \zeta^k \Upsilon_k(z), \quad \check{\Upsilon}^{[n]}(z, \zeta) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\zeta^k} \bar{\Upsilon}_k(z), \quad \zeta := \frac{v^2}{v^\perp}, \quad (3.14)$$

in terms of an infinite sequence of ordinary superfields  $\{\Upsilon_k(z)\}_{k=0}^{\infty}$  and their complex conjugates. The spinor covariant derivative  $\mathcal{D}_\alpha^{(1)}$  can be represented as

$$\mathcal{D}_\alpha^{(1)} = (v^\perp) \mathcal{D}_\alpha^{[1]}(\zeta), \quad \mathcal{D}_\alpha^{[1]}(\zeta) = \mathcal{D}_\alpha^2 - \zeta \mathcal{D}_\alpha^\perp. \quad (3.15)$$

From this representation, and the representation of the arctic multiplet in the north chart (3.14), it follows that the analyticity condition (3.8) nontrivially relates the superfield

coefficients  $\Upsilon_k(z)$  in the series.

Another important example not of the polar type and mentioned later is the smile-real *tropical multiplet*. A weight-0, real, tropical superfield  $V^{(0)}(z, v) = V^{[0]}(z, \zeta) = \sum_{k=-\infty}^{\infty} \zeta^k V_k(z)$  is required to be well-defined only on  $\mathbb{C}^*$ , that is,  $\mathbb{CP}^1$  with both north and south poles removed. The reality condition  $V^{(0)} = \check{V}^{(0)}$  implies that  $\overline{V_k} = (-1)^k V_{-k}$ . A special case of this is given by the product of a weight-0 arctic field and its smile-conjugate  $V^{(0)} \sim \check{\Upsilon}^{(0)} \Upsilon^{(0)}$ . A more detailed classification of 6D covariant projective superfields will be considered elsewhere. (See [46, 47] for a discussion in the flat case. In particular, it is shown in [47] how the flat six-dimensional vector multiplet is described in terms of a prepotential tropical superfield.)

For applications, it is crucial that the analyticity constraint defining projective superfields can be solved in terms of unconstrained isotwistor superfields and an *analytic projection operator*. We introduce the fourth-order operator

$$\Delta^{(4)} := \left( \mathcal{D}^{(4)} - \frac{5i}{6} C^{(2)\gamma\delta} \mathcal{D}_{\gamma\delta}^{(2)} - 5i \mathcal{C}^{(3)\gamma} \mathcal{D}_{\gamma}^{(1)} - \frac{i}{4} (\mathcal{D}_{\gamma\delta}^{(2)} C^{(2)\gamma\delta}) + 3 C^{(2)\gamma\delta} C_{\gamma\delta}^{(2)} \right), \quad (3.16)$$

where

$$\mathcal{D}^{(4)} := -\frac{1}{96} \varepsilon^{\alpha\beta\gamma\delta} \mathcal{D}_{\alpha}^{(1)} \mathcal{D}_{\beta}^{(1)} \mathcal{D}_{\gamma}^{(1)} \mathcal{D}_{\delta}^{(1)}, \quad \mathcal{D}_{\alpha\beta}^{(2)} := \mathcal{D}_{[\alpha}^{(1)} \mathcal{D}_{\beta]}^{(1)} = -\mathcal{D}_{\beta\alpha}^{(2)}, \quad (3.17)$$

and

$$\varepsilon^{\alpha\beta\gamma\delta} C_{\gamma\delta}^{(2)} = \varepsilon^{\alpha\beta\gamma\delta} (\gamma^a)_{\gamma\delta} C_a^{(2)} = 2(\tilde{\gamma}^a)^{\alpha\beta} C_a^{(2)} = 2C^{(2)\alpha\beta}, \quad (3.18a)$$

$$\varepsilon^{\alpha\beta\gamma\delta} (\mathcal{D}_{\beta}^{(1)} C_{\gamma\delta}^{(2)}) = -12 \mathcal{C}^{(3)\alpha} = -12 \mathcal{C}^{\alpha ijk} v_i v_j v_k, \quad \mathcal{C}^{(3)\alpha} := -\frac{1}{12} \varepsilon^{\alpha\beta\gamma\delta} (\mathcal{D}_{\beta}^{(1)} C_{\gamma\delta}^{(2)}). \quad (3.18b)$$

The superfield  $\mathcal{C}_{ijk}^{\alpha}$  is the dimension- $\frac{3}{2}$  torsion component defined in (2.15a). The crucial property of the analytic projection operator is that, given an arbitrary weight- $(n-4)$  isotwistor, Lorentz scalar superfield  $U^{(n-4)}$ , the superfield  $Q^{(n)}$  defined by

$$Q^{(n)}(z, v) := \Delta^{(4)} U^{(n-4)}(z, v), \quad (3.19)$$

is a weight- $n$  projective superfield:

$$\mathcal{D}_{\alpha}^{(1)} Q^{(n)} = 0. \quad (3.20)$$

Moreover, both  $Q^{(n)}$  and  $U^{(n-4)}$  can be required to transform homogeneously with respect to super-Weyl transformations in which case the transformations are fixed to be

$$\delta_{\sigma} U^{(n-4)} = (2n-2) \sigma U^{(n-4)}, \quad \delta_{\sigma} Q^{(n)} = 2n \sigma Q^{(n)}. \quad (3.21)$$

It is worth noting that the analytic projection operator can be also used to build a weight-4 projective multiplet  $\mathcal{P}^{(4)}(z, v)$  from a scalar,  $v$ -independent superfield  $P(z)$ . In fact, for any  $P(z)$ , the superfield  $\mathcal{P}^{(4)}(z, v) := \Delta^{(4)}P(z)$  is a weight-4 projective superfield. Moreover, if one wants both  $P$  and  $\mathcal{P}^{(4)}$  to transform homogeneously under super-Weyl transformations then they have to satisfy:  $\delta_\sigma P = 6\sigma P$  and  $\delta_\sigma \mathcal{P}^{(4)} = 8\sigma \mathcal{P}^{(4)}$ . A derivation of these statements is given in Appendix B. We conclude this subsection by giving the analytic projection operator in an equivalent form:<sup>5</sup>

$$\Delta^{(4)} = \varepsilon^{\alpha\beta\gamma\delta} \left( -\frac{1}{96} \mathcal{D}_\alpha^{(1)} \mathcal{D}_\beta^{(1)} \mathcal{D}_\gamma^{(1)} \mathcal{D}_\delta^{(1)} - \frac{5i}{12} \mathcal{D}_\alpha^{(1)} C_{\beta\gamma}^{(2)} \mathcal{D}_\delta^{(1)} - \frac{i}{8} (\mathcal{D}_{\alpha\beta}^{(2)} C_{\gamma\delta}^{(2)}) + \frac{3}{2} C_{\alpha\beta}^{(2)} C_{\gamma\delta}^{(2)} \right). \quad (3.22)$$

This expression will be useful in the next subsection.

## 3.2 The Action Principle

In this subsection, we give a projective superfield action principle invariant under the supergravity gauge group and super-Weyl transformations and such that, in the flat limit, it reduces to the six-dimensional action of [46, 47]. The latter is an extension of the one originally introduced in four dimensions in [1] and reformulated in a projective-invariant form in [67]. The result is a simple generalization of the action principle in the curved projective superspaces in  $2 \leq D \leq 5$  [10, 11, 12, 13, 14, 15].

Let  $\mathcal{L}^{(2)}$  be a real projective multiplet of weight-2. We assume that  $\mathcal{L}^{(2)}$  possesses the super-Weyl transformation

$$\delta_\sigma \mathcal{L}^{(2)} = 4\sigma \mathcal{L}^{(2)}, \quad (3.23)$$

which complies with the rule (3.9). We also introduce a real isotwistor superfield  $\Theta^{(-4)}$  such that

$$\delta_\sigma \Theta^{(-4)} = -2\sigma \Theta^{(-4)}, \quad \Delta^{(4)} \Theta^{(-4)} = 1. \quad (3.24)$$

Associated with  $\mathcal{L}^{(2)}$  and  $\Theta^{(-4)}$  is the following functional

$$S = \frac{1}{2\pi} \oint_C (v, dv) \int d^6x d^8\theta E \Theta^{(-4)} \mathcal{L}^{(2)}, \quad E^{-1} = \text{Ber}(E_A^M). \quad (3.25)$$

---

<sup>5</sup>It is instructive to compare the six-dimensional analytic projection operator with the five-dimensional one of [10, 11]. There, the projector was presented in the gauge  $C_{\hat{a}}^{ij} = 0$  ( $\hat{a} = 0, \dots, 4$  is the 5D vector index in the notation of [10]) with only the 5D scalar torsion  $S^{ij}$  appearing in the projector. With an appropriate dimensional truncation, one can see that the coefficients in the 6D and 5D projectors match.

This functional is invariant under arbitrary re-scalings  $v^i(t) \rightarrow c(t) v^i(t)$ ,  $\forall c(t) \in \mathbb{C}^*$ , where  $t$  denotes the evolution parameter along the integration contour. By using that under super-Weyl transformations,  $E$  transforms as

$$\delta_\sigma E = -2\sigma E \quad (3.26)$$

and the transformation properties (3.23)–(3.24), we find that the functional  $S$  is super-Weyl invariant. The action (3.25) is also invariant under arbitrary local supergravity gauge transformations (2.9), (2.10) and (3.3a). The invariance under general coordinate and Lorentz transformations is trivial given that both  $\Theta^{(-4)}$  and  $\mathcal{L}^{(2)}$  are Lorentz scalars. The invariance under the  $SU(2)$  transformations can be proved similarly to the  $2 \leq D \leq 5$  cases: First, we note that

$$U^{(-2)} := \Theta^{(-4)} \mathcal{L}^{(2)} \quad (3.27)$$

is a isotwistor multiplet of weight  $-2$ . Then, one verifies that

$$K^{ij} J_{ij} U^{(-2)} = -\partial^{(-2)} \left( K^{(2)} U^{(-2)} \right), \quad \partial^{(-2)} := \frac{1}{(v, u)} u^i \frac{\partial}{\partial v^i}. \quad (3.28)$$

Next, since  $K^{(2)} U^{(-2)}$  has weight zero, it is easy to see that

$$(v, dv) K^{ij} J_{ij} U^{(-2)} = -dt \frac{d}{dt} \left( K^{(2)} U^{(-2)} \right), \quad (3.29)$$

where, again,  $t$  denotes the evolution parameter along the integration contour in (3.25). Since the integration contour is closed, the  $SU(2)$ -part of the transformations of  $U^{(-2)}$  (3.3a) does not contribute to the variation of the action (3.25).

The isotwistor superfield  $\Theta^{(-4)}$  is used to ensure the invariance of the action under super-Weyl and  $SU(2)$  transformations. An important point is that, in general, the supersymmetric action (3.25) is independent of the explicit form of  $\Theta^{(-4)}$ , which is just an auxiliary constructive tool. To prove this, we need one observation about the analytic projection operator  $\Delta^{(4)}$  (3.16) or (3.22). Specifically, let us show that  $\Delta^{(4)}$  is symmetric under integration-by-parts. In the geometry of section 2, the rule for integration-by-parts is

$$\int d^6 x d^8 \theta E \mathcal{D}_A V^A = 0, \quad (3.30)$$

with  $V^A = (V^a, V_i^\alpha)$  an arbitrary superfield. Introducing arbitrary isotwistor superfields  $\Psi^{(-n)}$  and  $\Phi^{(n-6)}$ , and by using the form of the analytic projection operator given in

(3.22), we find the symmetry relation

$$\frac{1}{2\pi} \oint_C (v, dv) \int d^6x d^8\theta E \left\{ \Psi^{(-n)} \Delta^{(4)} \Phi^{(n-6)} - \Phi^{(n-6)} \Delta^{(4)} \Psi^{(-n)} \right\} = 0 . \quad (3.31)$$

Now, let  $\mathcal{U}^{(-2)}$  be a real isotwistor prepotential for the Lagrangian  $\mathcal{L}^{(2)}$  in (3.25):

$$\mathcal{L}^{(2)} = \Delta^{(4)} \mathcal{U}^{(-2)} . \quad (3.32)$$

By using (3.31) and  $\Delta^{(4)} \Theta^{(-4)} = 1$ , we can re-express the action (3.25) in the form

$$S = \frac{1}{2\pi} \oint_C (v, dv) \int d^6x d^8\theta E \mathcal{U}^{(-2)} . \quad (3.33)$$

If the Lagrangian  $\mathcal{L}^{(2)}$ , and hence  $\mathcal{U}^{(-2)}$ , is independent of  $\Theta^{(-4)}$  then (3.33) makes manifest the independence of (3.25) on  $\Theta^{(-4)}$ .

We point out that there is a freedom in the choice of  $\Theta^{(-4)}$ . For instance, given a real weight- $m$  isotwistor superfield  $\Gamma^{(m)}$ ,  $\Theta^{(-4)}$  may be defined as

$$\Theta^{(-4)} := \frac{\Gamma^{(m)}}{\Delta^{(4)} \Gamma^{(m)}} , \quad \delta_\sigma \Gamma^{(m)} = 2(m+3) \sigma \Gamma^{(m)} . \quad (3.34)$$

Additionally, one can consider a real Lorentz scalar and SU(2) invariant superfield  $P = P(z)$  such that

$$\Theta^{(-4)} := \frac{P}{\Delta^{(4)} P} ; \quad \delta_\sigma P = 6\sigma P . \quad (3.35)$$

Note that the use of  $P$  is inequivalent to that of a general, weight-0 isotwistor superfield  $\Gamma^{(0)}$  which may have non-trivial dependence on the projective parameter  $\zeta$  and is, as such, not invariant under SU(2) transformations.

Let us take the flat limit of the action (3.25). This, up to total flat vector derivatives, can be written as

$$\begin{aligned} S &= \frac{1}{2\pi} \oint_C (v, dv) \int d^6x d^8\theta \check{\Theta}^{(-4)} L^{(2)} = \frac{1}{2\pi} \oint_C (v, dv) \int d^6x D^{(-4)} D^{(4)} \check{\Theta}^{(-4)} L^{(2)} \Big|_{\theta=0} \\ &= \frac{1}{2\pi} \oint_C (v, dv) \int d^6x D^{(-4)} L^{(2)} \Big|_{\theta=0} , \end{aligned} \quad (3.36)$$

with  $L^{(2)}$ ,  $\check{\Theta}^{(-4)}$ , and  $D^{(4)}$  the flat-superspace limit of the Lagrangian  $\mathcal{L}^{(2)}$ , the density  $\Theta^{(-4)}$ , and the analytic projector  $\Delta^{(4)}$  (3.16), respectively. Here, we have also introduced the operator

$$D^{(-4)} := -\frac{1}{96} \varepsilon^{\alpha\beta\gamma\delta} D_\alpha^{(-1)} D_\beta^{(-1)} D_\gamma^{(-1)} D_\delta^{(-1)} , \quad D_\alpha^{(-1)} := \frac{u_i}{(v, u)} D_\alpha^i . \quad (3.37)$$

The flat action is invariant under arbitrary projective transformations of the form:

$$(u_i, v_i) \rightarrow (u_i, v_i) R, \quad R = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \text{GL}(2, \mathbb{C}). \quad (3.38)$$

As it is explicitly independent of  $u$ , the same invariance holds for the curved-superspace action (3.25). This invariance is a powerful tool in superspace theories with eight supercharges. For example, in 5D,  $\mathcal{N} = 1$  [10] and 4D,  $\mathcal{N} = 2$  [68] supergravity it has been used to reduce the projective action principle to components. Clearly, the same techniques can be used in the six-dimensional case to reduce the action (3.25).

One can rewrite the contour integral in the north chart of  $\mathbb{CP}^1$ ,  $v^\perp \neq 0$ , in terms of the inhomogeneous complex variable  $\zeta$

$$v^i = v^\perp \zeta^i, \quad \zeta^i = (1, \zeta), \quad \zeta_i = \varepsilon_{ij} \zeta^j = (-\zeta, 1), \quad \zeta = \frac{v^2}{v^\perp} \in \mathbb{C}. \quad (3.39)$$

The Lagrangian  $L^{(2)}(z, v)$  in the north chart can be rewritten as

$$L^{(2)}(z, v) := i(v^\perp)^2 \zeta L(z, \zeta). \quad (3.40)$$

Since the action and the Lagrangian are independent of  $u_i$ , we can make the conventional choice

$$u_i = (1, 0), \quad u^i = \varepsilon^{ij} u_j = (0, -1). \quad (3.41)$$

The action (3.36) is, then, rewritten as

$$S = \oint_C \frac{d\zeta}{2\pi i} \int d^6x \zeta (D^\perp)^4 L \Big|_{\theta=0}, \quad (D^\perp)^4 := -\frac{1}{96} \varepsilon^{\alpha\beta\gamma\delta} D_\alpha^\perp D_\beta^\perp D_\gamma^\perp D_\delta^\perp. \quad (3.42)$$

This expression is the rigid supersymmetric action in the 6D,  $\mathcal{N} = (1, 0)$  projective superspace of [46, 47]. Thus, our curved projective action principle is, as expected, a generalization of the known flat one.

### 3.3 Some Matter Systems

We conclude this section with examples of supergravity-matter systems. We start by considering two classes of projective superfield conformal compensators: an  $\mathcal{O}(2)$  multiplet, given by the real, linear superfield  $G^{(2)} := G^{ij} v_i v_j$  and a weight-1, arctic multiplet  $\Upsilon^{(1)}$  and its smile conjugate  $\check{\Upsilon}^{(1)}$  that describes the off-shell, charged hypermultiplet.

Note that to use the linear multiplet as a proper compensator,  $G^{ij}$  should be nowhere-vanishing which is equivalent to  $G := \sqrt{G^{ij}G_{ij}} \neq 0$ . This composite scalar and SU(2) invariant superfield, which has scale dimension 4,  $\delta G = 4\sigma G$ , can be used to choose the super-Weyl gauge  $G = 1$ . In this gauge,  $\mathcal{D}_\alpha^i G = \mathcal{D}_\alpha^i 1 = 0$  which, together with the analyticity constraint  $\mathcal{D}_\alpha^{(i} G^{jk)} = 0$ , implies that  $G^{ij} = w^{ij}$  is covariantly constant  $\mathcal{D}_\alpha^i w^{jk} = 0$  wherefore also the SU(2) group is broken to the U(1) subgroup that leaves  $w^{ij}$  invariant. By imposing the consistency of the supergravity algebra with the covariant constancy of  $w^{ij}$ ,  $\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} w^{kl} = 0$ , one can easily see that, in this gauge, the dimension-1 torsions satisfy<sup>6</sup>

$$N_{abc} = 0, \quad C_a^{ij} = C_a w^{ij}. \quad (3.43)$$

The Lagrangian for the  $\mathcal{O}(2)$  multiplet compensator is given by

$$\mathcal{L}_{\text{SG-linear}}^{(2)} = -G^{(2)} \ln \frac{G^{(2)}}{\text{i} \check{\Upsilon}^{(1)} \Upsilon^{(1)}}. \quad (3.44)$$

It encodes the dynamics of a massless improved linear multiplet coupled to conformal supergravity. It has the same form as the 4D,  $\mathcal{N} = 2$  counterpart given in [70] as a locally-supersymmetric extension of the projective-superspace formulation [1] for the 4D,  $\mathcal{N} = 2$  improved tensor multiplet [71, 72]. The action (3.44) is independent of the (ant-)arctic superfields  $\Upsilon^{(1)}, \check{\Upsilon}^{(1)}$  which turn out to be pure-gauge superfields. The Lagrangian (3.44) can be shown to be dual to the Lagrangian for an arctic compensator coupled to conformal supergravity:

$$\mathcal{L}_{\text{SG-hyper}}^{(2)} = -\text{i} \Upsilon^{(1)} \check{\Upsilon}^{(1)}. \quad (3.45)$$

The duality map is the same as in reference [70].

By using the compensators, we can couple supergravity to general matter. We consider a few examples which are familiar from the lower-dimensional cases; we refer the reader to [12, 70] for the geometric interpretation of the models that follow.

Consider a system of interacting weight-1 arctic multiplets  $\Upsilon^{(1)I}(v)$  and their smile-conjugates  $\check{\Upsilon}^{(1)\bar{I}}(v)$  described by a Lagrangian of the form [17]

$$\mathcal{L}_{\text{NLSM-conf}}^{(2)} = \text{i} K(\Upsilon^{(1)I}, \check{\Upsilon}^{(1)\bar{J}}). \quad (3.46)$$

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<sup>6</sup>Similar gauges in superspace were used before in 4D in [68, 69] and in 3D in [15].

Here,  $K(\Phi^I, \bar{\Phi}^{\bar{J}})$  is a real function of  $n$  complex variables  $\Phi^I$ , with  $I = 1, \dots, n$ , that satisfies the homogeneity condition

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) . \quad (3.47)$$

This Lagrangian represents a conformal non-linear sigma-model as in [17].

Given a system of  $m$  weight-0 arctic multiplets  $\Xi^i$ ,  $i = 1, \dots, m$ , and the conformal compensator  $\Upsilon^{(1)}$ , one can write the Lagrangian

$$\mathcal{L}_{\text{NLSM-hyper}}^{(2)} = \Upsilon^{(1)} \check{\Upsilon}^{(1)} \exp \left\{ \mathcal{K}(\Xi^i, \check{\Xi}^{\bar{j}}) \right\} . \quad (3.48)$$

The real function  $\mathcal{K}(\Xi^i, \check{\Xi}^{\bar{j}})$  can be interpreted as a Kähler potential since the Lagrangian is invariant under the transformation

$$\Upsilon^{(1)} \mapsto e^{-\Lambda(\Xi)} \Upsilon^{(1)} , \quad \mathcal{K}(\Xi, \check{\Xi}) \mapsto \mathcal{K}(\Xi, \check{\Xi}) + \Lambda(\Xi) + \bar{\Lambda}(\check{\Xi}) , \quad (3.49)$$

with  $\Lambda$  a holomorphic function. In the dual picture, where the compensator is given by a linear superfield, the previous Lagrangian is equivalent to

$$\mathcal{L}_{\text{NLSM-linear}}^{(2)} = G^{(2)} \mathcal{K}(\Xi^i, \check{\Xi}^{\bar{j}}) . \quad (3.50)$$

Next, we consider a system of  $n$  linear  $\mathcal{O}(2)$  multiplets  $G_I^{(2)}$ ,  $I = 1, \dots, n$ , coupled to conformal supergravity. The Lagrangian takes form

$$\mathcal{L}_{\text{SM-linear}}^{(2)} = \mathcal{L}(G_I^{(2)}) , \quad (3.51)$$

where  $\mathcal{L}$  is a real homogeneous function of degree-1:

$$G_I^{(2)} \frac{\partial}{\partial G_I^{(2)}} \mathcal{L} = \mathcal{L} . \quad (3.52)$$

More generally, it is possible to couple linear  $\mathcal{O}(2)$  multiplets and hypermultiplets in an arbitrary way provided that the Lagrangian  $\mathcal{L}^{(2)}(G^{(2)}, \Upsilon^{(1)}, \check{\Upsilon}^{(1)}, \Xi, \check{\Xi})$  is weight-2 in the sense that  $\mathcal{L}^{(2)}(c^2 G^{(2)}, c \Upsilon^{(1)}, c \check{\Upsilon}^{(1)}, \Xi, \check{\Xi}) = c^2 \mathcal{L}^{(2)}(G^{(2)}, \Upsilon^{(1)}, \check{\Upsilon}^{(1)}, \Xi, \check{\Xi})$  with  $c \in \mathbb{C}^*$ .

We conclude by considering some composite, weight-2, scaling-dimension-4, real projective superfields built from tensors and vector field-strengths. We begin by taking two tensor multiplets in the representations  $\Phi$  and  $V_i^\alpha$ , introduced in section 2.3, and coupling them through the composite  $\mathcal{O}(2)$  superfield

$$\mathcal{G}^{(2)} := i(\mathcal{D}_\alpha^{(1)} \Phi) V^{\alpha(1)} + \frac{i}{4} \Phi \mathcal{D}_\alpha^{(1)} V^{\alpha(1)} , \quad V^{\alpha(1)} := v_i V^{\alpha i} . \quad (3.53)$$



That this combination is analytic follows from a short calculation:

$$\begin{aligned}\mathcal{D}_\beta^{(1)}\mathcal{G}^{(2)} &= i(\mathcal{D}_\beta^{(1)}\mathcal{D}_\alpha^{(1)}\Phi)V^{\alpha(1)} - i(\mathcal{D}_\alpha^{(1)}\Phi)\mathcal{D}_\beta^{(1)}V^{\alpha(1)} + \frac{i}{4}(\mathcal{D}_\beta^{(1)}\Phi)\mathcal{D}_\alpha^{(1)}V^{\alpha(1)} + \frac{i}{4}\Phi\mathcal{D}_\beta^{(1)}\mathcal{D}_\alpha^{(1)}V^{\alpha(1)} \\ &= 4C_{\beta\alpha}^{(2)}\Phi V^{\alpha(1)} + \frac{i}{4}\Phi\mathcal{D}_\beta^{(1)}\mathcal{D}_\alpha^{(1)}V^{\alpha(1)} = 0.\end{aligned}\tag{3.54}$$

Here, we are using the constraints (2.27) and (2.29) in the second equality. The third equality uses  $\mathcal{D}_\beta^{(1)}\mathcal{D}_\alpha^{(1)}V^{\alpha(1)} = 16iC_{\beta\alpha}^{(2)}V^{\alpha(1)}$ , which follows from the tensor constraint (2.29). Additionally, it is non-trivial but easy to check that this composite field is a super-Weyl tensor of scaling dimension 4:  $\delta\mathcal{G}^{(2)} = 4\sigma\mathcal{G}^{(2)}$ . Of course, all the previous arguments also hold in the case that the two tensor multiplets are not independent one of one another but satisfy  $\Phi = \mathcal{D}_{\alpha i}V^{\alpha i}$  as in (2.30).

Comparison of the constraints to those of the vector multiplet (2.33) shows that the same arguments work if we formally replace the tensor potential  $V_i^\alpha \rightarrow F_i^\alpha$  with the vector field-strength. Thus, the coupling of a vector and a tensor multiplet naturally gives rise to the weight-2 projective composite superfield [29]

$$\mathcal{F}^{(2)} := i(\mathcal{D}_\alpha^{(1)}\Phi)F^{\alpha(1)} + \frac{i}{4}\Phi\mathcal{D}_\alpha^{(1)}F^{\alpha(1)}.\tag{3.55}$$

If one, furthermore, considers a vector multiplet prepotential, which can be shown to be described by a weight-0, real, tropical superfield  $\mathbf{V} := V^{(0)}$ , then it is possible to construct the Lagrangian

$$\mathcal{L}^{(2)} = \mathbf{V}\mathcal{F}^{(2)}.\tag{3.56}$$

This should be compared with the five-dimensional vector multiplet Lagrangian coupled to supergravity [10, 11].

Finally, we point out that we can further extend the construction of the previous bilinear: Consider a real weight-0 isotwistor superfield  $\Phi^{(0)}(z, v)$  and a real weight-1 isotwistor superfield  $\mathbf{V}^{\alpha(1)}$  constrained by

$$((\tilde{\gamma}_a)^{\alpha\beta}\mathcal{D}_\alpha^{(1)}\mathcal{D}_\beta^{(1)} + 16iC_a^{(2)})\Phi^{(0)}(z, v) = 0, \quad \delta_\sigma\Phi^{(0)} = 2\sigma\Phi^{(0)},\tag{3.57a}$$

$$\mathcal{D}_\alpha^{(1)}\mathbf{V}^{\beta(1)} - \frac{1}{4}\delta_\alpha^\beta\mathcal{D}_\alpha^{(1)}\mathbf{V}^{\alpha(1)} = 0, \quad \delta_\sigma\mathbf{V}^{\alpha(1)} = \frac{3}{2}\sigma\mathbf{V}^{\alpha(1)}.\tag{3.57b}$$

Then, analogously to the previous cases, the composite superfield

$$\mathcal{L}^{(2)} := i(\mathcal{D}_\alpha^{(1)}\Phi^{(0)})\mathbf{V}^{\alpha(1)} + \frac{i}{4}\Phi^{(0)}\mathcal{D}_\alpha^{(1)}\mathbf{V}^{\alpha(1)},\tag{3.58}$$

is a real, weight-2 projective superfield such that  $\delta_\sigma \mathcal{L}^{(2)} = 4\sigma \mathcal{L}^{(2)}$ . Note that, in this case,  $\mathcal{L}^{(2)}$  is not an  $\mathcal{O}(2)$  multiplet.

The Lagrangian (3.58) appears to be the projective superspace analogue of the harmonic superspace Lagrangian introduced by Sokatchev to describe an off-shell tensor multiplet [31]. The latter was constructed by first taking a tensor multiplet of the  $\Phi$ -type and an independent tensor multiplet of the  $V^{\alpha i}$ -type and lifting them to harmonic superspace by allowing them arbitrary dependence on the harmonics. The construction of the projective action (3.58) is analogous: We started with two copies of the tensor multiplet (in different representations) and took them off-shell by allowing them to have arbitrary dependence on the isotwistor variable  $v^i$ .

## 4 Conclusion

In this paper, we have initiated the study of six-dimensional,  $\mathcal{N} = (1, 0)$  supergravity in projective superspace. Beginning with the conventional constraints (2.13a)–(2.13c), we provided the solution (2.14a)–(2.17) of the Bianchi identities up to and including dimension- $\frac{3}{2}$ . Super-Weyl transformations (2.18a, 2.18b) preserving this geometry were computed and used to recover the components of the type-I Weyl multiplet of 6D,  $\mathcal{N} = (1, 0)$  conformal supergravity. Coupling this multiplet to a closed super-3-form, we recovered the type-II Weyl multiplet. With the supergeometry understood, projective isotwistor variables were introduced and used to define projective superfields. The defining constraint of such fields was solved by constructing the analytic projection operator (3.16), which was subsequently used to define a projective superspace action principle (3.25). This was checked to be invariant under super-Weyl, local super-Poincaré, and local  $SU(2)$  transformations and reduced to its flat limit which agrees with the flat actions of [46, 47]. We concluded with the presentation of families of examples of such action principles for supergravity-matter systems.

Clearly, much remains to be done to complete our understanding of  $\mathcal{N} = (1, 0)$  supergravity in six-dimensional projective superspace. Perhaps the most pressing open problem is the construction of the projective superspace analogue of Sokatchev’s harmonic supergravity [31]. This construction exploits a remarkable combination of harmonic superspace prepotentials, both representations of the tensor multiplet, and the dynamical equations of the “matter” components of the type-I multiplet to avoid multiplet doubling. The

Lagrangian (3.58) is similar to the doubled-tensor compensator Lagrangian central to that construction. It would be of interest to confirm their equivalence and work out the detailed relation between these constructions.

Additional directions of study include compactification to five dimensions and comparison with the work of [10, 11] and the recovery of our geometry from the gauging of the six-dimensional superconformal group along the lines of references [73, 74] which develop a direct link between superspace and superconformal tensor calculus.<sup>7</sup> More straightforward work in need of completion includes: the presentation of the complete solution of the Bianchi identities for the supergeometry of section 2; the investigation of six-dimensional supersymmetric backgrounds and projective superspace matter couplings as in the research on 4D and 5D anti-de-Sitter supergeometries [9, 75, 76, 77, 78]; a more systematic classification of covariant projective superfields in six dimensions; and the component reduction of the 6D projective action principle, for example along the lines of [10, 68], which, within our formalism, is a first step towards the analysis of various supergravity-matter systems in components (see *e. g.* [79]).

Finally, we mention that new results on the construction of higher-derivative supergravity actions in six-dimensions have been obtained in [40]. It would be interesting to understand how classes of higher-derivative actions are constructed in six-dimensional projective superspace.

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<sup>7</sup>An early attempt in six dimensions was made in reference [48].

## A Six-dimensional Notation and Conventions

Our six-dimensional superspace conventions are obtained by lifting the five-dimensional conventions established in references [80, 9, 10]. The procedure is to first define  $\gamma_a := -\Gamma_a C^{-1}$  and  $\tilde{\gamma}_a = -C\Gamma_a$  for  $a = 0, \dots, 3; 5$ . Then we take  $\gamma_6 = C^{-1}$  and  $\tilde{\gamma}_6 = -C$ .<sup>8</sup> The relative sign has been chosen so that the six  $8 \times 8$  Dirac matrices satisfy

$$\{\Gamma_a, \Gamma_b\} = -2\eta_{ab}\mathbf{1} , \quad (\text{A.1})$$

with  $a = 0, \dots, 3; 5, 6$  and

$$(\eta_{ab}) = \text{diag}(-1, 1, 1, 1, 1, 1) . \quad (\text{A.2})$$

The overall sign is chosen so that, in terms of explicit indices, the formulae are

$$\begin{aligned} (\gamma^a)_{\alpha\beta} &= (\Gamma^a)_{\alpha\beta} , & (\tilde{\gamma}^a)^{\alpha\beta} &= -(\Gamma^a)^{\alpha\beta} \quad \text{for } a = 0, 1, 2, 3; 5 \\ (\gamma_6)_{\alpha\beta} &= \varepsilon_{\alpha\beta} , & (\tilde{\gamma}_6)^{\alpha\beta} &= -\varepsilon^{\alpha\beta} . \end{aligned} \quad (\text{A.3})$$

In terms of Pauli-type matrices, the Dirac matrices take the form

$$\Gamma_a = \begin{pmatrix} 0 & (\gamma_a)_{\alpha\beta} \\ (\tilde{\gamma}_a)^{\beta\alpha} & 0 \end{pmatrix} \quad (\text{A.4})$$

with  $\alpha = 1, \dots, 4$ . We can give an explicit representation of  $\gamma_a, \tilde{\gamma}_a$  in terms of the 4D Pauli matrices. In particular, denoting the 4D,  $\text{SL}(2, \mathbb{C})$  spinor indices by  $\underline{\alpha} = 1, 2$  and  $\underline{\dot{\alpha}} = 1, 2$ , we use the representation

$$\gamma_a = \begin{pmatrix} 0 & -(\sigma_a)_{\underline{\alpha}\underline{\dot{\beta}}} \\ (\tilde{\sigma}_a)^{\underline{\dot{\alpha}}\underline{\beta}} & 0 \end{pmatrix} \quad (\text{A.5})$$

for  $a = 0, \dots, 3$  and

$$\gamma_5 = \begin{pmatrix} i\varepsilon_{\underline{\alpha}\underline{\beta}} & 0 \\ 0 & i\varepsilon^{\underline{\dot{\alpha}}\underline{\dot{\beta}}} \end{pmatrix} , \quad \gamma_6 = \begin{pmatrix} \varepsilon_{\underline{\alpha}\underline{\beta}} & 0 \\ 0 & -\varepsilon^{\underline{\dot{\alpha}}\underline{\dot{\beta}}} \end{pmatrix} \quad (\text{A.6})$$

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<sup>8</sup> Keeping this procedure in mind, it is easier to verify certain statements using formulae from five dimensions. For example, since  $\gamma_a^T = (C^{-1})^T \Gamma_a^T = -C^{-1} C \Gamma_a C^{-1} = -\gamma_a$  for  $a = 0, 1, \dots, 3; 5$  we need inspect only  $\gamma_6$  to conclude that these matrices are anti-symmetric.

and similarly for matrices with upper indices. They obey the Pauli-type algebra

$$\begin{aligned} (\gamma^a)_{\alpha\beta}(\tilde{\gamma}^b)^{\beta\gamma} + (\gamma^b)_{\alpha\beta}(\tilde{\gamma}^a)^{\beta\gamma} &= -2\eta^{ab}\delta_\alpha^\gamma, \\ (\tilde{\gamma}^a)^{\alpha\beta}(\gamma^b)_{\beta\gamma} + (\tilde{\gamma}^b)^{\alpha\beta}(\gamma^a)_{\beta\gamma} &= -2\eta^{ab}\delta_\gamma^\alpha. \end{aligned} \quad (\text{A.7})$$

Due to our sign choices, the five-dimensional subalgebra agrees with that of references [80, 9, 10].

Note that the six-dimensional Pauli-type matrices are antisymmetric

$$(\gamma_a)_{\alpha\beta} = -(\gamma_a)_{\beta\alpha}, \quad (\text{A.8})$$

implying an isomorphism between the space of six-dimensional vectors and antisymmetric  $4 \times 4$  spin matrices

$$V_{\alpha\beta} := (\gamma^a)_{\alpha\beta}V_a = -V_{\beta\alpha} \Leftrightarrow V_a = \frac{1}{4}(\tilde{\gamma}_a)^{\alpha\beta}V_{\alpha\beta}. \quad (\text{A.9})$$

The second relation is a consequence of the analysis below and equation (A.15) in particular. Similarly, six-dimensional 2-forms are in one-to-one correspondence with traceless  $4 \times 4$  matrices and (anti-)self-dual 3-forms are in correspondence with symmetric rank-2 spin matrices with their indices (down) up as we now work out in detail.

To begin, it is useful to define the normalized anti-symmetrized products of Pauli-type matrices

$$\begin{aligned} \gamma_{ab} &:= \gamma_{[a}\tilde{\gamma}_{b]} := \frac{1}{2}(\gamma_a\tilde{\gamma}_b - \gamma_b\tilde{\gamma}_a), & \gamma_{abc} &:= \gamma_{[a}\tilde{\gamma}_b\gamma_{c]} := \frac{1}{3!}(\gamma_a\tilde{\gamma}_b\gamma_c \pm \text{perm.}), \\ \tilde{\gamma}_{ab} &:= \tilde{\gamma}_{[a}\gamma_{b]} := \frac{1}{2}(\tilde{\gamma}_a\gamma_b - \tilde{\gamma}_b\gamma_a), & \tilde{\gamma}_{abc} &:= \tilde{\gamma}_{[a}\gamma_b\tilde{\gamma}_{c]} := \frac{1}{3!}(\tilde{\gamma}_a\gamma_b\tilde{\gamma}_c \pm \text{perm.}). \end{aligned} \quad (\text{A.10})$$

For example, this normalization implies

$$\gamma^{ab}\gamma^c = \gamma^{abc} + 2\eta^{c[a}\gamma^{b]}, \quad \tilde{\gamma}^c\gamma^{ab} = \tilde{\gamma}^{abc} - 2\eta^{c[a}\tilde{\gamma}^{b]}. \quad (\text{A.11})$$

On the other hand, a more commonly used convention regarding the 2-form matrix is as the spinor representation (2.7) of the Lorentz generator  $M_{ab}$  which is related by

$$(\Sigma^{ab})_\alpha{}^\beta = -\frac{1}{2}(\gamma^{ab})_\alpha{}^\beta. \quad (\text{A.12})$$

In terms of these matrices, we define

$$F_\alpha{}^\beta := \frac{1}{2}(\Sigma^{ab})_\alpha{}^\beta F_{ab} \Rightarrow F_{ab} = -(\Sigma_{ab})_\beta{}^\alpha F_\alpha{}^\beta. \quad (\text{A.13})$$

The second relation is a consequence of (A.17) which follows from the analysis below. Using the second type of matrix, we can construct  $\tilde{F}^\alpha{}_\beta := (\tilde{\gamma}^{ab})^\alpha{}_\beta F_{ab}$  but  $(\tilde{\gamma}^{ab})^\alpha{}_\beta = -(\gamma^{ab})_\beta{}^\alpha$  so that this second matrix is not essentially new.

Finally, the third-rank antisymmetric tensors can be separated into (anti-)self-dual parts which are then in one-to-one correspondence with symmetric  $4 \times 4$  matrices. To see how this works in detail, we must first establish some Fierz identities. There is a completeness relation

$$\frac{1}{2}(\gamma^a)_{\alpha\beta}(\gamma_a)_{\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta} . \quad (\text{A.14})$$

Contraction with  $\varepsilon^{\gamma'\delta'\gamma\delta}$  implies the completeness relation

$$\frac{1}{2}(\gamma^a)_{\alpha\beta}(\tilde{\gamma}_a)^{\gamma\delta} = \delta_\alpha^\gamma \delta_\beta^\delta - \delta_\beta^\gamma \delta_\alpha^\delta \quad (\text{A.15})$$

and that

$$\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}(\gamma_a)_{\gamma\delta} = (\tilde{\gamma}_a)^{\alpha\beta} \Rightarrow (\gamma_a)_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}(\tilde{\gamma}_a)^{\gamma\delta} . \quad (\text{A.16})$$

Contraction of (A.15) with itself gives

$$\frac{1}{4}(\tilde{\gamma}^{ab})^\alpha{}_\beta(\gamma_{ab})_\gamma{}^\delta = -\frac{1}{2}\delta_\beta^\alpha \delta_\gamma^\delta + 2\delta_\beta^\delta \delta_\gamma^\alpha . \quad (\text{A.17})$$

Another contraction with (A.15) gives

$$(\tilde{\gamma}^{abc})^{\alpha\beta}(\gamma_{abc})_{\gamma\delta} = 24(\delta_\gamma^\alpha \delta_\delta^\beta + \delta_\delta^\alpha \delta_\gamma^\beta) , \quad (\text{A.18})$$

while contraction with (A.14) shows that

$$(\gamma^{abc})_{\alpha\beta}(\gamma_{abc})_{\gamma\delta} = 0 \text{ and } (\tilde{\gamma}^{abc})^{\alpha\beta}(\tilde{\gamma}_{abc})^{\gamma\delta} = 0 . \quad (\text{A.19})$$

Thus we see that  $\tilde{\gamma}^{abc}$  and  $\gamma^{abc}$  correspond to (anti-)self-dual 3-forms. To show that  $(\gamma^{abc})$   $\tilde{\gamma}^{abc}$  is (anti-)self-dual, one can use the identities

$$\gamma^0 \tilde{\gamma}^1 \gamma^2 \tilde{\gamma}^3 \gamma^5 \tilde{\gamma}^6 = -\mathbf{1} \quad \text{and} \quad \tilde{\gamma}^0 \gamma^1 \tilde{\gamma}^2 \gamma^3 \tilde{\gamma}^5 \gamma^6 = +\mathbf{1} , \quad (\text{A.20})$$

to conclude that, for example,  $\gamma^{012} = -\epsilon^{012}{}_{345} \gamma^{345}$  whereas  $\tilde{\gamma}^{012} = \epsilon^{012}{}_{345} \tilde{\gamma}^{345}$ . Therefore, to conform to the accepted conventions on (anti-)self-duality, we normalize  $\epsilon^{012356} = 1$ .

From the trace relation on the 3-forms

$$\text{tr}(\tilde{\gamma}_{abc} \gamma^{def}) = 4! \left( \delta_{[a}^d \delta_b^e \delta_{c]}^f + \frac{1}{3!} \epsilon_{abc}{}^{def} \right) , \quad (\text{A.21})$$

it follows that the (anti-)self-dual parts of a 3-form  $N$  satisfy

$$\begin{aligned} N_{\alpha\beta} &:= \frac{1}{3!} N_{abc} (\gamma^{abc})_{\alpha\beta} \Rightarrow N_{abc}^{(+)} = \frac{1}{8} N_{\alpha\beta} (\tilde{\gamma}_{abc})^{\alpha\beta} , \\ N^{\alpha\beta} &:= \frac{1}{3!} N_{abc} (\tilde{\gamma}^{abc})^{\alpha\beta} \Rightarrow N_{abc}^{(-)} = \frac{1}{8} N^{\alpha\beta} (\gamma_{abc})_{\alpha\beta} . \end{aligned} \quad (\text{A.22})$$

In six dimensions, Hodge duality on 3-forms is an involution of order 2:

$$\frac{1}{3!} \epsilon_{abc rst} \epsilon^{def rst} = -3! \delta_{[a}^d \delta_b^e \delta_{c]}^f . \quad (\text{A.23})$$

Following [80], the components of the  $\text{SU}^*(4)$  spinor and its complex conjugate

$$(\Psi_\alpha) = \begin{pmatrix} \psi_{\underline{\alpha}} \\ \bar{\phi}^{\underline{\alpha}} \end{pmatrix} \quad \text{and} \quad (\Psi_{\bar{\alpha}}^*) = \begin{pmatrix} \bar{\psi}_{\dot{\underline{\alpha}}} \\ \phi^{\underline{\alpha}} \end{pmatrix} \quad (\text{A.24})$$

are defined in terms of 4D  $\text{SL}(2, \mathbb{C})$  spinors. Introducing the unitary matrix

$$(B_\alpha^{\bar{\beta}}) = \begin{pmatrix} 0 & \varepsilon_{\underline{\alpha}\underline{\beta}} \\ -\varepsilon_{\dot{\underline{\alpha}}\underline{\beta}} & 0 \end{pmatrix} \Rightarrow (B^{\text{T}\bar{\beta}}_\alpha) = \begin{pmatrix} 0 & \varepsilon_{\underline{\beta}\underline{\alpha}} \\ -\varepsilon_{\underline{\beta}\dot{\underline{\alpha}}} & 0 \end{pmatrix} , \quad (\text{A.25})$$

it can be checked explicitly that, using the representation defined by (A.5) and (A.6),

$$B_\alpha^{\bar{\alpha}} B_\beta^{\bar{\beta}} (\gamma_a^*)_{\bar{\alpha}\bar{\beta}} = (\gamma_a)_{\alpha\beta} . \quad (\text{A.26})$$

We may, therefore, define the covariant conjugate<sup>9</sup>

$$\overline{(\Psi_\alpha)} := (B_\alpha^{\bar{\alpha}} \Psi_{\bar{\alpha}}^*) = \begin{pmatrix} \phi_{\underline{\alpha}} \\ -\bar{\psi}^{\underline{\alpha}} \end{pmatrix} . \quad (\text{A.27})$$

The complex conjugate  $(B_{\bar{\alpha}}^{*\beta})$  satisfies

$$BB^* = -\mathbf{1} = B^*B . \quad (\text{A.28})$$

This implies that performing the conjugation twice,

$$\overline{\overline{\Psi}} = \overline{(B\Psi^*)} = B(B^*\Psi) = -\Psi . \quad (\text{A.29})$$

Finally, we define the doublet

$$(\Psi_\alpha^i) \quad \text{such that} \quad \Psi_\alpha^1 = \Psi_\alpha \quad \text{and} \quad \Psi_\alpha^2 = \bar{\Psi}_\alpha . \quad (\text{A.30})$$

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<sup>9</sup>The signs have been chosen so that the covariant conjugate reduces in five dimensions to the (transpose of the) Dirac conjugate in the conventions of [80].

This combination satisfies the SU(2)-Majorana-Weyl reality condition

$$\overline{(\Psi_\alpha^i)} = B_\alpha^{\bar{\alpha}} ((\Psi^i)^*)_{\bar{\alpha}} = \Psi_{\alpha i} , \quad (\text{A.31})$$

where we used the normalization  $\varepsilon_{\underline{21}} = 1$ . It follows from this that

$$\overline{\overline{\Psi^i}} = \Psi^i . \quad (\text{A.32})$$

Finally, the following conjugation relation holds

$$\overline{(\mathcal{D}_\alpha^i \Phi)} = -(-1)^{|\Phi|} \mathcal{D}_{\alpha i} \bar{\Phi} . \quad (\text{A.33})$$

## B On the Analytic Projector

In this appendix we derive the results stated at the end of section 3 about the analytic projection operator  $\Delta^{(4)}$  equations (3.16) and (3.22).

First of all we, want to prove that, given a general scalar weight- $n$  isotwistor superfield  $U^{(n)}$ , the superfield  $Q^{(n+4)} := \Delta^{(4)} U^{(n)}$  satisfies  $\mathcal{D}_\alpha^{(1)} Q^{(n+4)} = 0$ . The derivation goes along the same lines of the 5D case of [9, 10, 11].<sup>10</sup>

The starting point is to observe that by construction

$$\mathcal{D}_{[\alpha}^{(1)} \mathcal{D}_\beta^{(1)} \mathcal{D}_\gamma^{(1)} \mathcal{D}_\delta^{(1)} \mathcal{D}_{\rho]}^{(1)} = 0 , \quad (\text{B.1})$$

which implies

$$\begin{aligned} \mathcal{D}_\alpha^{(1)} \mathcal{D}^{(4)} = & -\frac{1}{480} \varepsilon^{\beta\gamma\delta\rho} \left( 4 \{ \mathcal{D}_\alpha^{(1)}, \mathcal{D}_\beta^{(1)} \} \mathcal{D}_\gamma^{(1)} \mathcal{D}_\delta^{(1)} \mathcal{D}_\rho^{(1)} + 3 \mathcal{D}_\gamma^{(1)} \{ \mathcal{D}_\alpha^{(1)}, \mathcal{D}_\beta^{(1)} \} \mathcal{D}_\delta^{(1)} \mathcal{D}_\rho^{(1)} \right. \\ & \left. + 2 \mathcal{D}_\gamma^{(1)} \mathcal{D}_\delta^{(1)} \{ \mathcal{D}_\alpha^{(1)}, \mathcal{D}_\beta^{(1)} \} \mathcal{D}_\rho^{(1)} + \mathcal{D}_\gamma^{(1)} \mathcal{D}_\delta^{(1)} \mathcal{D}_\rho^{(1)} \{ \mathcal{D}_\alpha^{(1)}, \mathcal{D}_\beta^{(1)} \} \right) . \end{aligned} \quad (\text{B.2})$$

Then, one applies the previous equation on the superfield  $U^{(n)}$  and compute the anti-commutators. Since  $[J^{(2)}, \mathcal{D}_\alpha^{(1)}] = J^{(2)} U^{(n)} = 0$ , the SU(2) part of the anti-commutator algebra (3.5) does not contribute at all in the computation. On the other hand, from the Lorentz part of (3.5) we need to systematically move to the right the Lorentz generator by

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<sup>10</sup>The analysis in this section generalize and in principle can be used to derive the analytic projection operator for the conformal supergravity geometry of [11] in the case where the 5D  $C_a^{ij}$  torsion component is nonzero. Such case was not presented in [11].



using  $[M_\alpha^\beta, \mathcal{D}_{\gamma k}] = (\frac{1}{4}\delta_\alpha^\beta \mathcal{D}_{\gamma k} - \delta_\gamma^\beta \mathcal{D}_{\alpha k})$  to then hit  $U^{(n)}$  and use  $M_\alpha^\beta U^{(n)} = 0$ . Moreover, one needs to elaborate on relations involving multiple  $\mathcal{D}^{(1)}$  derivatives of  $C_{\alpha\beta}^{(2)}$ . It is easy to prove that

$$\{\mathcal{D}_\alpha^{(1)}, \mathcal{D}_\beta^{(1)}\} C_{\gamma\delta}^{(2)} = 0 \quad \Leftrightarrow \quad \mathcal{D}_\alpha^{(1)} \mathcal{D}_\beta^{(1)} C_{\gamma\delta}^{(2)} = -\mathcal{D}_\beta^{(1)} \mathcal{D}_\alpha^{(1)} C_{\gamma\delta}^{(2)} = \mathcal{D}_{\alpha\beta}^{(2)} C_{\gamma\delta}^{(2)} . \quad (\text{B.3})$$

On the other hand, due to (2.16), we know that

$$\mathcal{D}_\alpha^{(1)} C_{\beta\gamma}^{(2)} = \mathcal{D}_{[\alpha}^{(1)} C_{\beta\gamma]}^{(2)} , \quad \mathcal{D}_\alpha^{(1)} C_{\beta\gamma}^{(2)} = 2\varepsilon_{\alpha\beta\gamma\delta} \mathcal{C}^{(3)\delta} . \quad (\text{B.4})$$

It is then clear that the following equation holds

$$\mathcal{D}_\alpha^{(1)} \mathcal{D}_\beta^{(1)} C_{\gamma\delta}^{(2)} = \mathcal{D}_{[\alpha}^{(1)} \mathcal{D}_{\beta}^{(1)} C_{\gamma\delta]}^{(2)} = \frac{1}{12} \varepsilon_{\alpha\beta\gamma\delta} (\mathcal{D}_{\rho\tau}^{(2)} C^{(2)\rho\tau}) . \quad (\text{B.5})$$

One can also derive this further result

$$\mathcal{D}_\alpha^{(1)} \mathcal{D}_{\beta\gamma}^{(2)} C_{\delta\rho}^{(2)} = -12i \mathcal{D}_\alpha^{(1)} C_{[\beta\gamma}^{(2)} C_{\delta\rho]}^{(2)} . \quad (\text{B.6})$$

At this point, after some algebra, one can obtain

$$\begin{aligned} \mathcal{D}_\alpha^{(1)} \mathcal{D}^{(4)} U^{(n)} &= \varepsilon^{\beta\gamma\delta\rho} \left( \frac{5i}{12} C_{\beta\gamma}^{(2)} \mathcal{D}_\alpha^{(1)} \mathcal{D}_{\delta\rho}^{(2)} - \frac{5i}{36} (\mathcal{D}_\beta^{(1)} C_{\gamma\delta}^{(2)}) \mathcal{D}_\alpha^{(1)} \mathcal{D}_\rho^{(1)} \right. \\ &\quad \left. + \frac{i}{48} (\mathcal{D}_{\beta\gamma}^{(2)} C_{\delta\rho}^{(2)}) \mathcal{D}_\alpha^{(1)} - \frac{3}{2} C_{\beta\gamma}^{(2)} C_{\delta\rho}^{(2)} \mathcal{D}_\alpha^{(1)} \right) U^{(n)} \end{aligned} \quad (\text{B.7})$$

It is then easy to get the following results

$$\mathcal{D}_\alpha^{(1)} (\varepsilon^{\beta\gamma\delta\rho} C_{\beta\gamma}^{(2)} \mathcal{D}_{\delta\rho}^{(2)} U^{(n)}) = \varepsilon^{\beta\gamma\delta\rho} \left( C_{\beta\gamma}^{(2)} \mathcal{D}_\alpha^{(1)} \mathcal{D}_{\delta\rho}^{(2)} + \frac{2}{3} (\mathcal{D}_\beta^{(1)} C_{\gamma\delta}^{(2)}) \mathcal{D}_\alpha^{(1)} \mathcal{D}_\rho^{(1)} \right) U^{(n)} , \quad (\text{B.8a})$$

$$\begin{aligned} \mathcal{D}_\alpha^{(1)} (\varepsilon^{\beta\gamma\delta\rho} (\mathcal{D}_\beta^{(1)} C_{\gamma\delta}^{(2)}) \mathcal{D}_\rho^{(1)} U^{(n)}) &= \varepsilon^{\beta\gamma\delta\rho} \left( -(\mathcal{D}_\beta^{(1)} C_{\gamma\delta}^{(2)}) \mathcal{D}_\alpha^{(1)} \mathcal{D}_\rho^{(1)} \right. \\ &\quad \left. - \frac{1}{4} (\mathcal{D}_{\beta\gamma}^{(2)} C_{\delta\rho}^{(2)}) \mathcal{D}_\alpha^{(1)} \right) U^{(n)} . \end{aligned} \quad (\text{B.8b})$$

$$\mathcal{D}_\alpha^{(1)} (\varepsilon^{\beta\gamma\delta\rho} (\mathcal{D}_\beta^{(1)} \mathcal{D}_\gamma^{(1)} C_{\delta\rho}^{(2)}) U^{(n)}) = \varepsilon^{\beta\gamma\delta\rho} \left( (\mathcal{D}_{\beta\gamma}^{(2)} C_{\delta\rho}^{(2)}) \mathcal{D}_\alpha^{(1)} - 12i (\mathcal{D}_\alpha^{(1)} C_{\beta\gamma}^{(2)} C_{\delta\rho}^{(2)}) \right) U^{(n)} , \quad (\text{B.8c})$$

$$\mathcal{D}_\alpha^{(1)} (\varepsilon^{\beta\gamma\delta\rho} (C_{\beta\gamma}^{(2)} C_{\delta\rho}^{(2)}) U^{(n)}) = \varepsilon^{\beta\gamma\delta\rho} \left( (C_{\beta\gamma}^{(2)} C_{\delta\rho}^{(2)}) \mathcal{D}_\alpha^{(1)} + (\mathcal{D}_\alpha^{(1)} C_{\beta\gamma}^{(2)} C_{\delta\rho}^{(2)}) \right) U^{(n)} . \quad (\text{B.8d})$$

By using the equations (B.7)–(B.8d) one can then observe that the combination of operators in the analytic projection operator (3.16) is such that

$$\mathcal{D}_\alpha^{(1)} \Delta^{(4)} U^{(n)} = 0 . \quad (\text{B.9})$$

As a next step we want to compute the super-Weyl transformations of the superfield  $Q^{(n+4)} = \Delta^{(4)}U^{(n)}$  supposing that  $U^{(n)}$  transforms homogeneously  $\delta_\sigma U^{(n)} = w\sigma U^{(n)}$ . To do that, we need some straightforward intermediate results. In particular, we have

$$\delta_\sigma \mathcal{D}_\alpha^{(1)} = \frac{1}{2}\sigma \mathcal{D}_\alpha^{(1)} - 2(\mathcal{D}_\beta^{(1)}\sigma)M_\alpha{}^\beta + 4(\mathcal{D}_\alpha^{(1)}\sigma)J^{(0)} - 4(\mathcal{D}_\alpha^{(-1)}\sigma)J^{(2)} , \quad (\text{B.10a})$$

$$\delta_\sigma C_{\alpha\beta}^{(2)} = \sigma C_{\alpha\beta}^{(2)} + \frac{i}{2}(\mathcal{D}_{\alpha\beta}^{(2)}\sigma) , \quad (\text{B.10b})$$

where

$$J^{(0)} := \frac{v_i u_j}{(v, u)} J^{ij} , \quad [J^{(0)}, \mathcal{D}_\gamma^{(1)}] = -\frac{1}{2}\mathcal{D}_\gamma^{(1)} , \quad J^{(0)}U^{(n)} = -\frac{n}{2}U^{(n)} , \quad (\text{B.11a})$$

$$\mathcal{D}_\alpha^{(-1)} := \frac{u_i}{(v, u)} \mathcal{D}_\alpha^i . \quad (\text{B.11b})$$

The  $(\mathcal{D}_\alpha^{(-1)}\sigma)$  term in (B.10a) does not actually enter into this computation since  $J^{(2)}$  commutes with the  $\mathcal{D}_\alpha^{(1)}$  derivatives and  $J^{(2)}U^{(n)} = 0$ . Defining,

$$\sigma_\alpha^{(1)} := \mathcal{D}_\alpha^{(1)}\sigma , \quad \sigma_{\alpha\beta}^{(2)} := \mathcal{D}_\alpha^{(1)}\mathcal{D}_\beta^{(1)}\sigma , \quad (\text{B.12a})$$

$$\sigma_{\alpha\beta\gamma}^{(3)} := \mathcal{D}_{[\alpha}^{(1)}\mathcal{D}_\beta^{(1)}\mathcal{D}_{\gamma]}^{(1)}\sigma , \quad \sigma_{\alpha\beta\gamma\delta}^{(4)} := \mathcal{D}_{[\alpha}^{(1)}\mathcal{D}_\beta^{(1)}\mathcal{D}_\gamma^{(1)}\mathcal{D}_{\delta]}^{(1)}\sigma , \quad (\text{B.12b})$$

we obtain other necessary intermediate results

$$\delta_\sigma \mathcal{C}^{(3)\alpha} = \frac{3}{2}\sigma \mathcal{C}^{(3)\alpha} + \frac{4}{3}\sigma_\beta^{(1)} C^{(2)\alpha\beta} - \frac{i}{24}\varepsilon^{\alpha\beta\gamma\delta}\sigma_{\beta\gamma\delta}^{(3)} , \quad (\text{B.13a})$$

$$\delta_\sigma \left( \mathcal{D}_{\gamma\delta}^{(2)} C^{(2)\gamma\delta} \right) = \left( 2\sigma (\mathcal{D}_{\gamma\delta}^{(2)} C^{(2)\gamma\delta}) + 120\sigma_\alpha^{(1)} \mathcal{C}^{(3)\alpha} - 8\sigma_{\gamma\delta}^{(2)} C^{(2)\gamma\delta} + \frac{i}{4}\varepsilon^{\alpha\beta\gamma\delta}\sigma_{\alpha\beta\gamma\delta}^{(4)} \right) . \quad (\text{B.13b})$$

Finally, after some algebra, one can obtain the following equation

$$\begin{aligned} \delta_\sigma \Delta^{(4)}U^{(n)} &= (w+2)\sigma \Delta^{(4)}U^{(n)} \\ &+ (w-2n-6) \left( -\frac{1}{24}\varepsilon^{\alpha\beta\gamma\delta}\sigma_\alpha^{(1)} \mathcal{D}_\beta^{(1)} \mathcal{D}_\gamma^{(1)} \mathcal{D}_\delta^{(1)} U^{(n)} - \frac{1}{16}\varepsilon^{\alpha\beta\gamma\delta}\sigma_{\alpha\beta}^{(2)} \mathcal{D}_{\gamma\delta}^{(2)} U^{(n)} \right. \\ &\quad - \frac{1}{24}\varepsilon^{\alpha\beta\gamma\delta}\sigma_{\alpha\beta\gamma}^{(3)} \mathcal{D}_\delta^{(1)} U^{(n)} - \frac{1}{96}\varepsilon^{\alpha\beta\gamma\delta}\sigma_{\alpha\beta\gamma\delta}^{(4)} U^{(n)} - \frac{5i}{3}\sigma_\alpha^{(1)} C^{(2)\alpha\beta} \mathcal{D}_\beta^{(1)} U^{(n)} \\ &\quad \left. - \frac{5i}{6}\sigma_{\alpha\beta}^{(2)} C^{(2)\alpha\beta} U^{(n)} + 5i\sigma_\alpha^{(1)} \mathcal{C}^{(3)\alpha} U^{(n)} \right) . \end{aligned} \quad (\text{B.14})$$

It is clear that by choosing

$$w = 2(n+3) , \quad (\text{B.15})$$

the weight- $(n+4)$  projective superfield  $Q^{(n+4)} = \Delta^{(4)}U^{(n)}$  transforms homogeneously,  $\delta_\sigma Q^{(n+4)} = 2(n+4)\sigma Q^{(n+4)}$ , in agreement with equation (3.9).

To conclude this appendix, we point out that, in the case  $n = 0$ , all the previous results are exactly the same if instead of a weight-0 isotwistor superfield  $U^{(0)}$  one considers a  $v$ -independent superfield  $P(z)$  such that  $\delta_\sigma P = 6\sigma P$ . Then, the superfield  $\mathcal{P}^{(4)} := \Delta^{(4)}P$  is a weight-4 projective superfield. To convince oneself of this, one has only to notice that for both  $U^{(0)}$  and  $P$  the conformal weight is 6 and that  $J^{(2)}U^{(0)} = J^{(0)}U^{(0)} = J^{(2)}P = J^{(0)}P = 0$  holds.

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